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Comments on Non-holomorphic Modular Forms and Non-compact Superconformal Field Theories

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Abstract

We extend our previous work [1] on the non-compact $\mathcal{N}=2$ $SCFT_2$ defined as the supersymmetric $SL(2,\mathbb{R})/U(1)$-gauged WZW model. Starting from path-integral calculations of torus partition functions of both the axial-type ('cigar') and the vector-type ('trumpet') models, we study general models of the $\mathbb{Z}_M$-orbifolds and M-fold covers with an arbitrary integer M. We then extract contributions of the degenerate representations ('discrete characters') in such a way that good modular properties are preserved. The 'modular completion' of the extended discrete characters introduced in [1] are found to play a central role as suitable building blocks in every model of orbifolds or covering spaces. We further examine a large M-limit (the 'continuum limit'), which 'deconstructs' the spectral flow orbits while keeping a suitable modular behavior. The discrete part of partition function as well as the elliptic genus is then expanded by the modular completions of *irreducible* discrete characters, which are parameterized by both continuous and discrete quantum numbers *modular transformed in a mixed way.* This limit is naturally identified with the universal cover of trumpet model. We finally discuss a classification of general modular invariants based on the modular completions of irreducible characters constructed above.

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1 Introduction

In this paper we try to extend our previous work [1] on the supersymmetric (SUSY) $SL(2,\mathbb{R})/U(1)$-gauged WZW model, that is, the SUSY non-linear σ-model on 2-dimensional black-hole [2]. In spite of its simplicity there are several intriguing features which originate from the non-compactness of target space. Among other things it would be surprising enough that this model could lead to a *non-holomorphic* elliptic genus [3, 1, 4], in other words, *would-be lack of holomophic factorization* in the torus partition function [1]. In more detail the torus partition function of this model is found to be expressed in the form as [1][1]

$$Z(\tau) = Z_{\mathbf{dis}}(\tau) + Z_{\mathbf{con}}(\tau). \tag{1.1}$$

The 'continuous part' $Z_{\mathbf{con}}(\tau)$ is mainly contributed from free strings propagating in the asymptotic region of 2D black-hole background, and is written in a holomorphically factorized form composed of characters of non-degenerate representations. On the other hand, the 'discrete part' $Z_{\mathbf{dis}}(\tau)$, which includes strings localized near the tip of 2D black-hole, would not be written in a holomorphically factorized form. It is modular invariant and formally expressible in an analogous way to rational conformal field theories (RCFTs);

$$Z_{\mathbf{dis}}(\tau) = \sum_{j,\tilde{j}} N_{j,\tilde{j}}\, \widehat{\chi}_j(\tau)\, \widehat{\chi}_{\tilde{j}}(\tau)^*.$$

However, the building blocks $\widehat{\chi}_j(\tau)$ are *no longer* holomorphic. They are written such as

$$\widehat{\chi}_j(\tau) = \chi_j(\tau) + [\text{subleading term, function of } \tau_2], \quad (\tau_2 \equiv \operatorname{Im}\tau), \tag{1.2}$$

where $\chi_j(\tau)$ denotes the 'extended discrete character' defined by spectral flow orbits of irreducible characters generated by BPS states [10, 7]. Although $\widehat{\chi}_j(\tau)$ shows the same IR behavior (around $\tau_2 \sim +\infty$) as $\chi_j(\tau)$, it is never expressible in terms only of characters of superconformal algebra due to the τ_2-dependence in the subleading term. We emphasize that, while the discrete characters $\chi_j(\tau)$ themselves are not, the functions $\widehat{\chi}_j(\tau)$ show simple modular behaviors mimicking RCFTs and are closed under modular transformations. When performing the S-transformation of $\chi_j(\tau)$, a continuous term of 'Mordell integral' [11, 12] emerges in the similar manner to the $\mathcal{N}=4$ characters [13]. However, the subleading term exactly cancels it out, simplifying considerably the modular transformation formulas of $\widehat{\chi}_j(\tau)$. Therefore, we shall call them the 'modular completions' of discrete characters.

Related mathematical studies of non-holomorphic modular and Jacobi forms [14] seem to have been still in new area. (See *e.g.* [15, 16].) Very roughly speaking, one finds correspondences

[1] Previous studies closely related to this subject have been given *e.g.* in [5, 6, 7, 8, 9].

such as

$$\text{discrete character} \longleftrightarrow \text{mock modular form (mock theta function)},$$
$$\text{modular completion} \longleftrightarrow \text{(harmonic) Maass form.}$$

Another possible application of the theory of mock modular forms to superconformal field theories has been presented in [17].

What we would like to clarify in this paper is addressed as follows;

(1) General $\mathbb{Z}_M$-orbifolds with arbitrary M :

For the parafermion theory $SU(2)/U(1)$ [18, 19], which is a 'compact analogue' of $SL(2)/U(1)$-theory, general modular invariants have been classified and interpreted as some orbifolds [19]. Inspired by this fact, we will examine general $\mathbb{Z}_M$-orbifolds of SUSY $SL(2)/U(1)$-coset with arbitrary M. This would be a natural extension of the analysis given in our previous work [1]. We especially would like to clarify the roles played by modular completions introduced in [1] in general models of orbifolds.

(2) Modular completion of *irreducible* discrete character :

In [1] we only introduced the modular completions of the extended discrete characters. It may be a natural question what is the modular completions of the *irreducible* discrete characters. We also would like to clarify the model including these new completions as natural building blocks.

This paper is organized as follows;

In section 2, we demonstrate the path-integral evaluations of torus partition functions of both axial-type ('cigar') and vector-type ('trumpet') SUSY $SL(2,\mathbb{R})/U(1)$ models, with an IR-regularization preserving good modular behaviors.

In section 3, we study general models of the $\mathbb{Z}_M$-orbifolds of cigar and the M-fold covers of trumpet with an arbitrary integer M, related with each other by the T-duality as expected [20]. We then extract contributions of the degenerate representations ('discrete characters'), which are captured by the modular completions of extended discrete characters. Especially, the discrete parts of partition functions of general orbifolds are defined so that the modular invariance is preserved. The 'twisted' discrete partition functions are also introduced and they show the modular covariance. We further discuss a 'continuum limit' by suitably taking $M \to \infty$, which leads us to the modular completion of *irreducible* discrete characters.

In section 4, we study the elliptic genera of relevant models. It will turn out that they are rewritten as linear combinations of the modular completions in all the cases.

In section 5, we discuss general forms of modular invariants when assuming the modular completions to be fundamental building blocks. Fourier transforms of the irreducible modular completions play a crucial role, and we will see that all the discrete partition functions and elliptic genera presented above are reexpressed in a unified way based on them.

We will summarize the main results and give some discussions in section 6.

2 Variants of $SL(2)/U(1)$ SUSY Coset Conformal Field Theories

2.1 Preliminaries : SUSY Gauged WZW Actions

We shall first introduce the model which we study in this paper, summarizing relevant notations. We consider the $SL(2,\mathbb{R})/U(1)$ SUSY gauged WZW model with level $k(>0)$[2], which is quite familiar [21] to have $\mathcal{N}=2$ superconformal symmetry with central charge;

$$\hat{c} \equiv \frac{c}{3} = 1 + \frac{2}{k}, \quad k \equiv \kappa - 2. \tag{2.1}$$

We restrict ourselves to cases with rational level $k = N/K$ ($N, K \in \mathbb{Z}_{>0}$) for the time being, and will later discuss the models with general levels allowed to be irrational. Note here that we do *not* necessarily assume that N and K are co-prime.

The world-sheet action of relevant SUSY gauged WZW model in the present convention is written as

$$\begin{aligned}
S(g, A, \psi^{\pm}, \tilde{\psi}^{\pm}) &:= \kappa S_{\mathrm{gWZW}}(g, A) + S_{\psi}(\psi^{\pm}, \tilde{\psi}^{\pm}, A), && (2.2)\\
\kappa S_{\mathrm{gWZW}}(g, A) &:= \kappa S_{\mathrm{WZW}}^{SL(2,\mathbb{R})}(g) + \frac{\kappa}{\pi}\int_{\Sigma} d^2v \, \Big\{ A_{\bar{v}} \mathrm{Tr}\left(\frac{\sigma_2}{2}\partial_v g g^{-1}\right) \pm \mathrm{Tr}\left(\frac{\sigma_2}{2} g^{-1}\partial_{\bar{v}} g\right) A_v \\
&\qquad \pm \mathrm{Tr}\left(\frac{\sigma_2}{2} g \frac{\sigma_2}{2} g^{-1}\right) A_{\bar{v}} A_v + \frac{1}{2} A_{\bar{v}} A_v \Big\}, && (2.3)\\
S_{\mathrm{WZW}}^{SL(2,\mathbb{R})}(g) &:= -\frac{1}{8\pi}\int_{\Sigma} d^2v \, \mathrm{Tr}\left(\partial_\alpha g^{-1} \partial_\alpha g\right) + \frac{i}{12\pi}\int_B \mathrm{Tr}\left((g^{-1}dg)^3\right), && (2.4)\\
S_{\psi}(\psi^{\pm}, \tilde{\psi}^{\pm}, A) &:= \frac{1}{2\pi}\int d^2v \, \Big\{ \psi^+(\partial_{\bar{v}} + A_{\bar{v}})\psi^- + \psi^-(\partial_{\bar{v}} - A_{\bar{v}})\psi^+ \\
&\qquad + \tilde{\psi}^+(\partial_v \pm A_v)\tilde{\psi}^- + \tilde{\psi}^-(\partial_v \mp A_v)\tilde{\psi}^+ \Big\}. && (2.5)
\end{aligned}$$

[2] k is the level of the total $SL(2,\mathbb{R})$-current including fermionic degrees of freedom, whose bosonic part has the level $\kappa \equiv k+2$.

In (2.3) and (2.5), the $+$ sign/ $-$ sign is chosen for the axial-like/vector-like gauged WZW model, which we shall denote as $S^{(A)}_{\text{gWZW}}/S^{(V)}_{\text{gWZW}}$ $(S^{(A)}_{\psi}/S^{(V)}_{\psi})$ from now on. The $U(1)$ chiral gauge transformation is defined by

$$
\begin{aligned}
& g \longmapsto \Omega_L\, g\, \Omega_R^{\epsilon}, \\
& A_{\bar{v}} \longmapsto A_{\bar{v}} - \Omega_L^{-1}\partial_{\bar{v}}\Omega_L, \quad A_v \longmapsto A_v - \Omega_R^{-1}\partial_v \Omega_R, \\
& \psi^{\pm} \longmapsto \Omega_L^{\pm 1}\, \psi^{\pm}, \quad \tilde{\psi}^{\pm} \longmapsto \Omega_R^{\pm\epsilon}\, \tilde{\psi}^{\pm}, \\
& \qquad\qquad \Omega_L(v,\bar{v}),\ \Omega_R(v,\bar{v}) \in e^{i\mathbb{R}\sigma_2},
\end{aligned} \tag{2.6}
$$

where we set $\epsilon \equiv +1, -1$ for the axial, vector model, respectively. The gauged WZW action $S^{(A)}_{\text{gWZW}}$ / $S^{(V)}_{\text{gWZW}}$ is invariant under the axial/vector type gauge transformations that correspond to $\Omega_L = \Omega_R$ in (2.6). Both of the classical fermion actions $S^{(A)}_{\psi}$, $S^{(V)}_{\psi}$ (2.5) are invariant under general chiral gauge transformations Ω_L, Ω_R, and we assume the absence of chiral anomalies when $\Omega_L = \Omega_R$ holds.

It is well-known that this model describes the string theory on 2D Euclidean black-hole [2]. The axial-type corresponds to the cigar geometry, while the vector-type does to the 'trumpet', which is T-dual to the cigar [20]. We will later elaborate their precise relation from the viewpoints of torus partition functions.

It will be convenient to introduce alternative notations of gauged WZW actions;

$$
S^{(A)}_{\text{gWZW}}(g, h_L, h_R) := S^{SL(2,\mathbb{R})}_{\text{WZW}}(h_L g h_R) - S^{SL(2,\mathbb{R})}_{\text{WZW}}(h_L h_R^{-1}), \tag{2.7}
$$

$$
S^{(V)}_{\text{gWZW}}(g, h_L, h_R) := S^{SL(2,\mathbb{R})}_{\text{WZW}}(h_L g h_R) - S^{SL(2,\mathbb{R})}_{\text{WZW}}(h_L h_R). \tag{2.8}
$$

They are indeed equivalent with (2.3) under the identification of gauge field;

$$
A_{\bar{v}} \frac{\sigma_2}{2} = \partial_{\bar{v}} h_L h_L^{-1}, \qquad A_v \frac{\sigma_2}{2} = \epsilon\, \partial_v h_R h_R^{-1}, \tag{2.9}
$$

where we set $\epsilon = +1$, (-1) for the axial (vector) model as before, as one can confirm by using the Polyakov-Wiegmann identity;

$$
S^{SL(2,\mathbb{R})}_{\text{WZW}}(gh) = S^{SL(2,\mathbb{R})}_{\text{WZW}}(g) + S^{SL(2,\mathbb{R})}_{\text{WZW}}(h) + \frac{1}{\pi}\int_{\Sigma} d^2v\, \mathrm{Tr}\left(g^{-1}\partial_{\bar{v}} g\, \partial_v h h^{-1}\right). \tag{2.10}
$$

2.2 Axial Coset : Euclidean Cigar

We shall first focus on the axial model. We are interested in the torus partition function. We define the world-sheet torus Σ by the identifications $(w, \bar{w}) \sim (w + 2\pi, \bar{w} + 2\pi) \sim (w + 2\pi\tau, \bar{w} +$

$2\pi\bar{\tau}$) ($\tau \equiv \tau_1 + i\tau_2$, $\tau_2 > 0$, and use the convention $v = e^{iw}$, $\bar{v} = e^{-i\bar{w}}$). We call the cycles defined by these two identifications as the α and β-cycles as usual.

Detailed calculations of the torus partition function have been carried out in [7, 1] based on the Wick rotated model (*i.e.* $H^3_+/\mathbb{R}_A$ supercoset, with $H^3_+ \equiv SL(2,\mathbb{C})/SU(2)$). Especially, the partition function of $\widetilde{\mathrm{R}}$-sector (R-sector with $(-1)^F$ insertion) with the $\mathcal{N}=2$ moduli z, $\bar{z}$ (*i.e.* the insertion of $e^{2\pi i\left(zJ_0 - \bar{z}\tilde{J}_0\right)}$, where J, $\tilde{J}$ are $\mathcal{N}=2$ $U(1)$-currents) has been presented in our previous work [1]. We shall just sketch it here.

In the Wick rotated model $H^3_+/\mathbb{R}_A$, the gauge field $A \equiv (A_{\bar{v}}d\bar{v} + A_v dv)\dfrac{\sigma_2}{2}$ should be regarded as a hermitian 1-form. Following the familiar treatment of gauged WZW models (see *e.g.* [22, 23, 24]), we decompose the gauge field as follows;

$$A[u]_{\bar{w}} = \partial_{\bar{w}}X + i\partial_{\bar{w}}Y - \frac{u}{2\tau_2}, \quad A[u]_w = \partial_w X - i\partial_w Y - \frac{\bar{u}}{2\tau_2} \tag{2.11}$$

where X, Y are real scalar fields parameterizing the chiral gauge transformations (in the Wick rotated model);

$$\Omega_L = e^{(X+iY)\frac{\sigma_2}{2}}, \quad \Omega_R\left(\equiv \Omega_L^\dagger\right) = e^{(X-iY)\frac{\sigma_2}{2}}, \tag{2.12}$$

and $u \equiv s_1\tau + s_2$, $(0 \leq s_1, s_2 < 1)$ is the modulus of gauge field. To emphasize the modulus dependence of gauge field we took the notation '$A[u]$'. Note that the modulus parameter u is normalized so that it correctly couples with the zero-modes of $U(1)$-currents J^3, $\tilde{J}^3$ which should be gauged;

$$-\frac{\partial}{\partial u}S(g, a[u], \psi^\pm, \tilde{\psi}^\pm)\Big|_{u=0} = 2\pi i J^3_0, \quad -\frac{\partial}{\partial \bar{u}}S(g, a[u], \psi^\pm, \tilde{\psi}^\pm)\Big|_{u=0} = -2\pi i \tilde{J}^3_0, \tag{2.13}$$

where we set $a[u] \equiv (a[u]_{\bar{w}}d\bar{w} + a[u]_w dw)\dfrac{\sigma_2}{2} \equiv \left(-\dfrac{u}{2\tau_2}d\bar{w} - \dfrac{\bar{u}}{2\tau_2}dw\right)\dfrac{\sigma_2}{2}$. For later convenience, we introduce the 'monodromy function';

$$\Phi[u](w,\bar{w}) = \frac{i}{2\tau_2}\left\{(\bar{w}\tau - w\bar{\tau})s_1 + (\bar{w} - w)s_2\right\} \equiv \frac{1}{\tau_2}\mathrm{Im}(w\bar{u}), \tag{2.14}$$

satisfying the twisted boundary conditions;

$$\Phi[u](w+2\pi, \bar{w}+2\pi) = \Phi[u](w,\bar{w}) - 2\pi s_1, \quad \Phi[u](w+2\pi\tau, \bar{w}+2\pi\bar{\tau}) = \Phi[u](w,\bar{w}) + 2\pi s_2. \tag{2.15}$$

We also introduce the notation;

$$h_u \equiv e^{i\Phi[u](w,\bar{w})\frac{\sigma_2}{2}}. \tag{2.16}$$

Then, the modulus part of gauge field is expressed as

$$a[u]_{\bar{w}} = i\partial_{\bar{w}}\Phi[u], \quad a[u]_w = -i\partial_w\Phi[u]. \tag{2.17}$$

Including the 'angle parameter' z which couples with the $U(1)_R$-symmetry in $\mathcal{N}=2$ superconfomral symmetry, the torus partition function is written as

$$
\begin{aligned}
Z(\tau,z) = & \int_\Sigma \frac{d^2u}{\tau_2} \int \mathcal{D}[g, A[u], \psi^\pm, \tilde{\psi}^\pm] \\
& \times \exp\left[-\kappa S^{(A)}_{\mathrm{gWZW}}\left(g, A[u+\frac{2}{k}z]\right) - S^{(A)}_\psi\left(\psi^\pm, \tilde{\psi}^\pm, A[u+\frac{k+2}{k}z]\right)\right], \qquad (2.18)
\end{aligned}
$$

where $\frac{d^2u}{\tau_2} \equiv ds_1 ds_2$ is the modular invariant measure of modulus parameter u, and we work in the $\widetilde{\mathrm{R}}$-sector for world-sheet fermions. We can explicitly evaluate this path-integration by separating the degrees of freedom of chiral gauge transformations (real scalar fields X and Y) according to the standard quantization of gauged WZW models [22, 23, 24], which renders this model 'almost' a free conformal system. Namely, interactions among each sector are caused only through the integration of modulus u. One can easily confirm that the complex parameter z precisely corresponds to the insertion of an operator $e^{2\pi i\left(zJ_0 - \bar{z}\tilde{J}_0\right)}$, where J and $\tilde{J}$ are the $\mathcal{N}=2$ $U(1)$-currents in the Kazama-Suzuki model [21]. (See [1] for more detail.)

To proceed further we have to path-integrate the compact boson Y, while the non-compact boson X is decoupled as a gauge volume. By using the definitions of (2.7), (2.8) and a suitable change of integration variables, we obtain

$$
\begin{aligned}
Z(\tau,z) \;=\; & \int_\Sigma \frac{d^2u}{\tau_2} \int \mathcal{D}[g, Y, \psi^\pm, \tilde{\psi}^\pm, b, \widetilde{b}, c, \widetilde{c}] \\
& \times \exp\left[-\kappa S^{(V)}_{\mathrm{gWZW}}\left(g, h^{u+\frac{2}{k}z}, \left(h^{u+\frac{2}{k}z}\right)^\dagger\right) + \kappa S^{(A)}_{\mathrm{gWZW}}\left(e^{iY\sigma_2}, h^{u+\frac{2}{k}z}, h^{u+\frac{2}{k}z}\right)\right] \\
& \times \exp\left[-2 S^{(A)}_{\mathrm{gWZW}}\left(e^{iY\sigma_2}, h^{u+\frac{k+2}{k}z}, h^{u+\frac{k+2}{k}z}\right) - S^{(A)}_\psi\left(\psi^\pm, \tilde{\psi}^\pm, a[u+\frac{k+2}{k}z]\right)\right] \\
& \times \exp\left[-S_{\mathrm{gh}}(b, \widetilde{b}, c, \widetilde{c})\right], \qquad (2.19)
\end{aligned}
$$

where the ghost variables have been introduced to rewrite the Jacobian factor. It is most non-trivial to evaluate the path-integration of the compact boson Y. Its world-sheet action is evaluated as

$$
\begin{aligned}
S^{(A)}_Y(Y,u,z) \;\equiv\; & -\kappa S^{(A)}_{\mathrm{gWZW}}\left(e^{iY\sigma_2}, h^{u+\frac{2}{k}z}, \left(h^{u+\frac{2}{k}z}\right)^\dagger\right) + 2 S^{(A)}_{\mathrm{gWZW}}\left(e^{iY\sigma_2}, h^{u+\frac{k+2}{k}z}, \left(h^{u+\frac{k+2}{k}z}\right)^\dagger\right) \\
=\; & \frac{k}{\pi}\int_\Sigma d^2v\, \partial_{\bar{w}} Y^u \partial_w Y^u - \frac{2\pi}{\tau_2}\hat{c}\,|z|^2, \qquad (2.20)
\end{aligned}
$$

where we set $Y^u \equiv Y + \Phi[u]$. Note that the linear couplings between the $U(1)$-currents $i\partial_w Y$, $i\partial_{\bar{w}} Y$ and the $\mathcal{N}=2$ moduli z, $\bar{z}$ are precisely canceled out[3]. Since Y^u satisfies the following

[3] Of course, this cancellation is expected by construction of the $\mathcal{N}=2$ superconformal algebras in the Kazama-Suzuki supercoset.

boundary condition;

$$\begin{aligned}
&Y^u(w+2\pi,\bar{w}+2\pi) = Y^u(w,\bar{w}) - 2\pi(m_1+s_1),\\
&Y^u(w+2\pi\tau,\bar{w}+2\pi\bar{\tau}) = Y^u(w,\bar{w}) + 2\pi(m_2+s_2), \quad (m_1,m_2\in\mathbb{Z}),
\end{aligned} \tag{2.21}$$

the zero-mode integral yields the summation over winding sectors weighted by the factor $e^{-\frac{\pi k}{\tau_2}|m_1\tau+m_2+u|^2} \equiv e^{-\frac{\pi k}{\tau_2}|(m_1+s_1)\tau+(m_2+s_2)|^2}$ determined by the 'instanton action'. After all, we achieve the next formula of partition function;

$$\begin{aligned}
Z(\tau,z) &= \mathcal{N}\, e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2-\frac{k+4}{k}z_2^2\right)} \sum_{m_1,m_2\in\mathbb{Z}} \int_\Sigma \frac{d^2u}{\tau_2} \left|\frac{\theta_1\left(\tau,u+\frac{k+2}{k}z\right)}{\theta_1\left(\tau,u+\frac{2}{k}z\right)}\right|^2 e^{-4\pi\frac{u_2z_2}{\tau_2}}\, e^{-\frac{\pi k}{\tau_2}|m_1\tau+m_2+u|^2}\\
&\equiv \mathcal{N}\, e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2-\frac{k+4}{k}z_2^2\right)} \int_{\mathbb{C}} \frac{d^2u}{\tau_2} \left|\frac{\theta_1\left(\tau,u+\frac{k+2}{k}z\right)}{\theta_1\left(\tau,u+\frac{2}{k}z\right)}\right|^2 e^{-4\pi\frac{u_2z_2}{\tau_2}}\, e^{-\frac{\pi k}{\tau_2}|u|^2},
\end{aligned} \tag{2.22}$$

where $\mathcal{N}$ is a normalization constant. This is identified as the Euclidean cigar model whose asymptotic circle has the radius $\sqrt{\alpha' k}$.

To be more precise, one should make a suitable regularization of (2.22), since it shows an IR divergence that originates from the non-compactness of target space. In other words, the integral of modulus u logarithmically diverges due to the quadratic behavior of integrand $\sim 1/\left|u+\frac{2}{k}z\right|^2$ near the point $u=-\frac{2}{k}z\in\Sigma$. According to [1], we take the regularization such that the integration region of modulus u is replaced with

$$\Sigma(z,\epsilon) \equiv \Sigma \setminus \left\{ u = s_1\tau+s_2 \;\; ; \quad -\frac{\epsilon}{2}-\frac{2}{k}\zeta_1 < s_1 < \frac{\epsilon}{2}-\frac{2}{k}\zeta_1, \;\; 0<s_2<1 \right\}, \tag{2.23}$$

where we set $z\equiv\zeta_1\tau+\zeta_2$, $\zeta_1,\zeta_2\in\mathbb{R}$, and $\epsilon(>0)$ denotes the regularization parameter. Then, the regularized partition function is defined as

$$Z_{\mathbf{reg}}(\tau,z;\epsilon) = \mathcal{N}\, e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2-\frac{k+4}{k}z_2^2\right)} \sum_{m_1,m_2\in\mathbb{Z}} \int_{\Sigma(z,\epsilon)} \frac{d^2u}{\tau_2} \left|\frac{\theta_1\left(\tau,u+\frac{k+2}{k}z\right)}{\theta_1\left(\tau,u+\frac{2}{k}z\right)}\right|^2 e^{-4\pi\frac{u_2z_2}{\tau_2}}\, e^{-\frac{\pi k}{\tau_2}|m_1\tau+m_2+u|^2}. \tag{2.24}$$

One of the main results of [1] is the 'character decomposition' of (2.24). Namely, it has been shown that the partition function can be uniquely decomposed in such a form as

$$Z_{\mathbf{reg}}(\tau,z;\epsilon) = [\text{sesquilinear form of } \widehat{\chi}_{\mathbf{dis}}(\tau,z)] + [\text{sesquilinear form of } \chi_{\mathbf{con}}(\tau,z)], \tag{2.25}$$

where $\widehat{\chi}_{\mathbf{dis}}(\tau,z)$ denotes the 'modular completion' of extended discrete character [1], while $\chi_{\mathbf{con}}(\tau,z)$ does the extended continuous character [10, 7] only attached with a real 'Liouville momentum' $p\in\mathbb{R}$ (above the 'mass gap', in other words). Their precise definitions and relevant formulas are summarized in Appendix C.

Important points are addressed as follows;

- The modular completion $\widehat{\chi}_{\mathbf{dis}}(\tau,z)$ is non-holomorphic with respect to modulus τ, but possesses simple modular properties: the S-transformation is closed by themselves, whereas the extended discrete character $\chi_{\mathbf{dis}}(\tau,z)$ is not.

- The second term in (2.25) shows a logarithmic divergence under the $\epsilon \to +0$ limit, which corresponds to the contribution from strings freely propagating in asymptotic region. On the other hand, the first term remains finite under $\epsilon \to +0$, and we denote it as $Z_{\mathbf{dis}}(\tau,z)$ (the 'discrete part' of partition function). $Z_{\mathbf{dis}}(\tau,z)$ is modular invariant by itself, as we will later elaborate on it.

- It is worth pointing out that the decomposition (2.25) itself *uniquely* determines the functional form of modular completion $\widehat{\chi}_{\mathbf{dis}}(\tau,z)$. In fact, the subleading terms in (1.2) are unambiguously determined by making 'completion of the square' for terms including the extended discrete characters $\chi_{\mathbf{dis}}(\tau,z)$ (C.15) in the decomposition of $Z_{\mathbf{reg}}$.

2.3 Vector Coset : Euclidean Trumpet

The partition function of the vector-type model is defined in the same way as (2.18), with the vector-type gauged WZW action $S^{(V)}_{\mathrm{gWZW}}(g,\tilde{A})$. In the Wick-rotated model, we should again regard as $g \in H^3_+$, while the gauge field $\tilde{A}$ is now parameterized as

$$\tilde{A}[u]_{\bar{w}} = \partial_{\bar{w}} X + i\partial_{\bar{w}} Y - \frac{u}{2\tau_2}, \quad \tilde{A}[u]_{w} = \partial_{w} X - i\partial_{w} Y + \frac{\bar{u}}{2\tau_2}. \tag{2.26}$$

Again, the non-compact direction X is anomaly free (vector-like), and the compact-direction Y is anomalous (axial-like). We should note that the gauge field $\tilde{A}$ is neither a hermitian nor an anti-hermitian 1-form. This fact originates from the sign difference of modulus $\bar{u}$ compared with (2.11), which has been chosen so that it leads to the same coupling to current zero-modes as given in (2.13).

Now, the wanted partition function is written as

$$\begin{aligned}\widetilde{Z}(\tau,z) = \int_{\Sigma} \frac{d^2u}{\tau_2} \int \mathcal{D}[g,\tilde{A}[u],\psi^{\pm},\tilde{\psi}^{\pm}] \\ \times \exp\left[-\kappa S^{(V)}_{\mathrm{gWZW}}\left(g,\tilde{A}[u+\frac{2}{k}z]\right) - S^{(V)}_{\psi}\left(\psi^{\pm},\tilde{\psi}^{\pm},\tilde{A}[u+\frac{k+2}{k}z]\right)\right]. \end{aligned}\tag{2.27}$$

We can again evaluate it by using the formulas (2.7) and (2.8) as follows[4];

$$
\begin{aligned}
\widetilde{Z}(\tau,z) \;=\;& \int_\Sigma \frac{d^2u}{\tau_2}\int \mathcal{D}[g,Y,\psi^\pm,\tilde{\psi}^\pm,b,\widetilde{b},c,\widetilde{c}] \\
&\times \exp\left[-\kappa S^{(V)}_{\mathrm{gWZW}}\left(g, h^{u+\frac{2}{k}z}, \left(h^{u+\frac{2}{k}z}\right)^\dagger\right) + \kappa S^{(V)}_{\mathrm{gWZW}}\left(e^{iY\sigma_2}, h^{u+\frac{2}{k}z}, \left(h^{u+\frac{2}{k}z}\right)^\dagger\right)\right] \\
&\times \exp\left[-2S^{(V)}_{\mathrm{gWZW}}\left(e^{iY\sigma_2}, h^{u+\frac{k+2}{k}z}, \left(h^{u+\frac{k+2}{k}z}\right)^\dagger\right) - S^{(A)}_\psi\left(\psi^\pm,\tilde{\psi}^\pm, a[u+\frac{k+2}{k}z]\right)\right] \\
&\times \exp\left[-S_{\mathrm{gh}}(b,\widetilde{b},c,\widetilde{c})\right]. \qquad (2.28)
\end{aligned}
$$

In deriving (2.28), we assumed that the path-integral measure of fermions is anomalous along the axial direction (Y) as opposed to the axial model (2.19).

The world-sheet action of compact boson Y is now evaluated as

$$
\begin{aligned}
S^{(V)}_Y(Y,u) \;\equiv\;& -\kappa S^{(V)}_{\mathrm{gWZW}}\left(e^{iY\sigma_2}, h^{u+\frac{2}{k}z}, \left(h^{u+\frac{2}{k}z}\right)^\dagger\right) + 2S^{(V)}_{\mathrm{gWZW}}\left(e^{iY\sigma_2}, h^{u+\frac{k+2}{k}z}, \left(h^{u+\frac{k+2}{k}z}\right)^\dagger\right) \\
=\;& \frac{k}{\pi}\int_\Sigma d^2v\, \partial_{\bar{w}}Y\partial_w Y - \frac{ik}{2\pi}\int_\Sigma d\Phi[u]\wedge dY. \qquad (2.29)
\end{aligned}
$$

Note that the z-dependence is completely canceled out contrary to the axial case (2.20). Moreover, the absence of quadratic term of modulus u is characteristic for the vector-type model. The second term in (2.29) is non-dynamical and contributes to the path-integral just through 'winding numbers';

$$
\int_\alpha d\Phi[u] = -2\pi s_1, \quad \int_\beta d\Phi[u] = -2\pi s_2, \quad \int_\alpha dY = 2\pi n_1, \quad \int_\beta dY = 2\pi n_2, \quad (n_1,n_2\in\mathbb{Z}).
$$

In this way we obtain

$$
Z^{(V)}_Y(\tau,u) = \frac{\sqrt{k}}{\sqrt{\tau_2}\,|\eta(\tau)|^2}\sum_{n_1,n_2\in\mathbb{Z}} e^{-\frac{\pi k}{\tau_2}|n_1\tau+n_2|^2}\, e^{-2\pi i k(s_1n_2-s_2n_1)}. \qquad (2.30)
$$

However, we face a subtlety since k is fractional in general. We recall $k=N/K$, and assume that N and K are coprime from now on. The periodicity of moduli parameters $s_i \to s_i+1$, $(i=1,2)$ would be violated unless $n_1,n_2\in K\mathbb{Z}$. In other words, one should impose this restriction of winding numbers to assure the consistency of functional integration.

[4] A caution: after separating chiral gauge transformations, the fermion action should get $S^{(A)}_\psi\left(\psi^\pm,\tilde{\psi}^\pm,a[*]\right)$, rather than $S^{(V)}_\psi\left(\psi^\pm,\tilde{\psi}^\pm,a[*]\right)$. It is due to our parameterization of gauge field $\tilde{A}$. (Recall how (2.26) includes the modulus $\bar{u}$.) This fact leads us to the correct fermion factor $\left|\theta_1\left(\tau,u+\frac{k+2}{k}\right)\right|^2$ in the partition function (2.31).

Combining all the pieces and by taking the regularization: $\Sigma \to \Sigma(\epsilon, z)$ (2.23), we finally achieve the following expression for the vector-type coset;

$$\begin{aligned}\widetilde{Z}_{\mathbf{reg}}(\tau, z; \epsilon) &= \mathcal{N}\, e^{-\frac{2\pi}{\tau_2}\frac{k+4}{k}z_2^2} \sum_{n_1, n_2 \in \mathbb{Z}} \int_{\Sigma(z,\epsilon)} \frac{d^2u}{\tau_2} \left| \frac{\theta_1\left(\tau, u + \frac{k+2}{k}z\right)}{\theta_1\left(\tau, u + \frac{2}{k}z\right)} \right|^2 \\ &\qquad \times e^{-4\pi\frac{u_2 z_2}{\tau_2}} e^{-\frac{\pi NK}{\tau_2}|n_1\tau + n_2|^2} e^{2\pi i N(n_1 s_2 - n_2 s_1)}. \end{aligned} \tag{2.31}$$

One can also make the character decomposition for (2.31). We will work on this subject in the next section. Before that, let us first discuss aspects of the $\mathbb{Z}_M$-orbifolds of $SL(2)/U(1)$ cosets with an arbitrary integer M systematically.

3 General $\mathbb{Z}_M$-Orbifold of $SL(2)/U(1)$

We next consider the general $\mathbb{Z}_M$-orbifold of $SL(2)/U(1)$-model. We again assume a model of rational level; $k = N/K$ ($N, K \in \mathbb{Z}_{>0}$) and let M be an arbitrary divisor of N, setting $N = ML$, $L \in \mathbb{Z}_{>0}$[5].

3.1 $\mathbb{Z}_M$-Orbifold and M-fold Cover

We start with the axial model. Since the twisted boson Y^u introduced in (2.20) is associated with the (asymptotic) angle coordinate of cigar geometry, one may consistently define the $\mathbb{Z}_M$-orbifold by introducing fractional winding sectors $m_1, m_2 \in \frac{1}{M}\mathbb{Z}$, leading to the torus partition function;

$$Z^{(M)}_{\mathbf{reg}}(\tau, z; \epsilon) = \frac{1}{M} \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_M} Z^{(M)}_{\mathbf{reg}}(\tau, z\,|\alpha_1, \alpha_2; \epsilon), \tag{3.1}$$

$$\begin{aligned} Z^{(M)}_{\mathbf{reg}}(\tau, z\,|\alpha_1, \alpha_2; \epsilon) &:= k\, e^{-\frac{2\pi}{\tau_2}\frac{k+4}{k}z_2^2} \sum_{m_1, m_2 \in \mathbb{Z}} \int_{\Sigma(z,\epsilon)} \frac{d^2u}{\tau_2} \\ &\quad \times \left| \frac{\theta_1\left(\tau, u + \frac{k+2}{k}z\right)}{\theta_1\left(\tau, u + \frac{2}{k}z\right)} \right|^2 e^{-4\pi\frac{u_2 z_2}{\tau_2}} e^{-\frac{\pi k}{\tau_2}\left|u + \left(m_1 + \frac{\alpha_1}{M}\right)\tau + \left(m_2 + \frac{\alpha_2}{M}\right)\right|^2}. \end{aligned} \tag{3.2}$$

Here we again took the regularization (2.24) and chose the normalization constant as $\mathcal{N} = k$ so that

$$\lim_{\epsilon \to +0} \lim_{z \to 0} Z^{(M)}_{\mathbf{reg}}(\tau, z\,|0, 0; \epsilon) = 1.$$

[5] Here we do *not* assume that N and K are coprime integers. Therefore, in case M is not a divisor of N, one may just replace N, K with $N' \equiv NM$, $K' \equiv KM$, and all the following arguments are applicable.

We also included an modular invariant factor $e^{-2\pi\frac{\hat{c}}{\tau_2}|z|^2}$ by hand to avoid unessential complexity of equations below[6].

The twisted partition function (3.2) behaves 'almost' modular covariantly;

$$\begin{aligned} & Z_{\mathbf{reg}}^{(M)}(\tau+1,z\,|\,\alpha_1,\alpha_2;\epsilon) = Z_{\mathbf{reg}}^{(M)}(\tau,z\,|\,\alpha_1,\alpha_1+\alpha_2;\epsilon), \\ & Z_{\mathbf{reg}}^{(M)}\left(-\frac{1}{\tau},\frac{z}{\tau}\,\middle|\,\alpha_1,\alpha_2;\epsilon\right) = Z_{\mathbf{reg}}^{(M)}(\tau,z\,|\,\alpha_2,-\alpha_1;\epsilon)+\mathcal{O}(\epsilon\log\epsilon), \end{aligned} \tag{3.3}$$

as is directly checked by the definition (3.2). Note that the violation of S-covariance in (3.3) is at most at the order of $\mathcal{O}(\epsilon\log\epsilon)$.

It is convenient to introduce the 'Fourier transform' of (3.2) by the next relations;

$$\widetilde{Z}_{\mathbf{reg}}^{(M)}(\tau,z\,|\,\beta_1,\beta_2;\epsilon) \;=\; \frac{1}{M}\sum_{\alpha_1,\alpha_2\in\mathbb{Z}_M} e^{2\pi i\frac{1}{M}(\alpha_1\beta_2-\alpha_2\beta_1)}\, Z_{\mathbf{reg}}^{(M)}(\tau,z\,|\,\alpha_1,\alpha_2;\epsilon), \tag{3.4}$$

Using the identity

$$\sum_{m_1,m_2\in\mathbb{Z}} e^{-\frac{\pi\alpha}{\tau_2}|(m_1+s_1)\tau+(m_2+s_2)|^2} e^{2\pi i(m_1t_2-m_2t_1)} = \frac{1}{\alpha}\sum_{n_1,n_2\in\mathbb{Z}} e^{-\frac{\pi}{\alpha\tau_2}|(n_1+t_1)\tau+(n_2+t_2)|^2} e^{2\pi i[(n_1+t_1)s_2-(n_2+t_2)s_1]},$$
$$(\mathrm{Re}\,\alpha>0,\quad s_i,t_i\in\mathbb{R}), \tag{3.5}$$

which is proven by the Poisson resummation formula, we obtain the explicit form of (3.4) as

$$\begin{aligned} \widetilde{Z}_{\mathbf{reg}}^{(M)}(\tau,z\,|\,\beta_1,\beta_2;\epsilon) \;=\; & M e^{-\frac{2\pi}{\tau_2}\frac{k+4}{k}z_2^2}\sum_{n_1,n_2\in\mathbb{Z}}\int_{\Sigma(z,\epsilon)}\frac{d^2u}{\tau_2}\, e^{-4\pi\frac{u_2z_2}{\tau_2}}\left|\frac{\theta_1\left(\tau,u+\frac{k+2}{k}z\right)}{\theta_1\left(\tau,u+\frac{2}{k}z\right)}\right|^2 \\ & \times e^{-\frac{\pi}{k\tau_2}|(Mn_1+\beta_1)\tau+(Mn_2+\beta_2)|^2}\, e^{2\pi i\{(Mn_2+\beta_2)s_1-(Mn_1+\beta_1)s_2\}}. \end{aligned} \tag{3.6}$$

It shows the same modular properties;

$$\begin{aligned} & \widetilde{Z}_{\mathbf{reg}}^{(M)}(\tau+1,z\,|\,\beta_1,\beta_2;\epsilon) = \widetilde{Z}_{\mathbf{reg}}^{(M)}(\tau,z\,|\,\beta_1,\beta_1+\beta_2;\epsilon), \\ & \widetilde{Z}_{\mathbf{reg}}^{(M)}\left(-\frac{1}{\tau},\frac{z}{\tau}\,\middle|\,\beta_1,\beta_2;\epsilon\right) = \widetilde{Z}_{\mathbf{reg}}^{(M)}(\tau,z\,|\,\beta_2,-\beta_1;\epsilon)+\mathcal{O}(\epsilon\log\epsilon). \end{aligned} \tag{3.7}$$

We here present some considerations about physical interpretations of partition functions;

1. From the definition (3.2) itself, it is obvious that $Z_{\mathbf{reg}}^{(M)}(\tau,z\,|\,0,0;\epsilon) = Z_{\mathbf{reg}}(\tau,z;\epsilon)$ (2.24) for an arbitrary M (up to the factor $e^{-2\pi\frac{\hat{c}}{\tau_2}|z|^2}$). This fact is not surprising, since $Z_{\mathbf{reg}}^{(M)}(\tau,z\,|\,0,0;\epsilon)$

[6] In convention adopted in this paper, the axial coset includes the factor $e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}$, while the vector coset does the different factor $e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2}\left(\equiv e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\cdot e^{-2\pi\frac{\hat{c}}{\tau_2}|z|^2}\right)$. We shall unify these 'anomaly factors' to the latter one just for convenience. Otherwise, one would be bothered about factors such as $e^{-2\pi\frac{\hat{c}}{\tau_2}|z|^2}$ *e.g.* in (3.4).

is associated to the untwisted sector of $\mathbb{Z}_M$-orbifold. It actually depends only on $k = N/K$, and is independent of the choice of pair N, K. As already mentioned, this is identified as the Euclidean cigar model with the asymptotic radius $\sqrt{\alpha' k}$. We also note that the cigar partition function (2.24) is rewritten in the 'T-dualized' form (with the factor $e^{-2\pi\frac{\hat{c}}{\tau_2}|z|^2}$ included);

$$\begin{aligned} Z_{\mathrm{cigar},\,\mathbf{reg}}(\tau, z; \epsilon) \left(\equiv Z^{(M)}_{\mathbf{reg}}(\tau, z\,|\,0,0;\epsilon)\right) &= e^{-\frac{2\pi}{\tau_2}\frac{k+4}{k}z_2^2} \sum_{n_1,n_2\in\mathbb{Z}} \int_{\Sigma(z,\epsilon)} \frac{d^2u}{\tau_2}\, e^{-4\pi\frac{u_2 z_2}{\tau_2}} \\ &\quad \times \left| \frac{\theta_1\left(\tau, u + \frac{k+2}{k}z\right)}{\theta_1\left(\tau, u+\frac{2}{k}z\right)} \right|^2 e^{-\frac{\pi}{k\tau_2}|n_1\tau+n_2|^2}\, e^{2\pi i(n_2 s_1 - n_1 s_2)}, \end{aligned} \quad (3.8)$$

by using the identity (3.5). We identify the R.H.S of (3.8) with the partition function of Euclidean trumpet whose asymptotic circle has the radius $\sqrt{\frac{\alpha'}{k}}$ [20]. In summary, we conclude

$$Z^{(M)}_{\mathbf{reg}}(\tau, z\,|\,0,0;\epsilon) \iff \text{cigar} \overset{\text{T-dual}}{\iff} \text{trumpet}, \quad (\text{indep. of } M). \quad (3.9)$$

2. The relation (3.4) tells us that $\widetilde{Z}^{(M)}_{\mathbf{reg}}(\tau, z\,|\,0,0;\epsilon)$ is identified with the $\mathbb{Z}_M$-orbifold of cigar. On the other hand, the expression (3.6) is naturally interpreted as the 'M-fold cover of trumpet', since the winding numbers are restricted to multiples of M if compared with (3.8). Namely, one can summarize that

$$\begin{aligned} \widetilde{Z}^{(M)}_{\mathbf{reg}}(\tau, z\,|\,0,0;\epsilon) &= \frac{1}{M}\sum_{\alpha_1,\alpha_2\in\mathbb{Z}_M} Z^{(M)}_{\mathbf{reg}}(\tau, z\,|\,\alpha_1,\alpha_2;\epsilon) \\ &\iff [\mathbb{Z}_M\text{-orbifold of cigar}] \\ &\overset{\text{T-dual}}{\iff} [M\text{-fold cover of trumpet }]. \end{aligned} \quad (3.10)$$

By comparing (2.31) with (3.6), we also find that the vector-type coset is identified with $\widetilde{Z}^{(N)}_{\mathbf{reg}}(\tau, z\,|\,0,0;\epsilon)$ *when N and K are coprime,* that is,

$$\begin{aligned} [\text{vector-type } SL(2)_{k=N/K}/U(1)] &\iff [\mathbb{Z}_N\text{-orbifold of cigar}] \\ &\overset{\text{T-dual}}{\iff} [N\text{-fold cover of trumpet }]. \end{aligned} \quad (3.11)$$

In other words, we also find the equivalence;

$$\begin{aligned} &[\text{vector-type } SL(2)_{k=N/K}/U(1)] \\ &\qquad \iff [\mathbb{Z}_N\text{-orbifold of axial-type } SL(2)_{k=N/K}/U(1)]. \end{aligned} \quad (3.12)$$

It is an obvious analogue of the familiar equivalence in parafermion theory - $SU(2)/U(1)$-coset (see *e.g.* [19, 25]);

$$[\text{vector-type } SU(2)_N/U(1)] \iff [\mathbb{Z}_N\text{-orbifold of axial-type } SU(2)_N/U(1)]. \quad (3.13)$$

3. In the above, we described the $\mathbb{Z}_M$-orbifolds of cigar and the M-fold covers of trumpet for an arbitrary integer M, which are T-dual with each other. One might then ask; *how about the M-fold covers of cigar?* However, they are not well-defined. A geometrically manifest reason is the fact that $\pi_1(\text{cigar}) = 0$ holds, whereas $\pi_1(\text{trumpet}) = \mathbb{Z}$. In other words, one cannot restrict $m_1, m_2 \in M\mathbb{Z}$, $(M > 1)$ in (2.24), without spoiling the expected periodicity of moduli $s_i \to s_i + n_i$, $({}^\forall n_i \in \mathbb{Z})$.

In the similar sense the vector type model (2.31) only allows $\mathbb{Z}_M$-orbifolds with a divisor M of N, while the M-fold covers are well-defined for arbitrarily large M. This is again because of the compatibility with periodicity of s_i.

3.2 Discrete Parts of Partition Functions

Let us focus on the discrete parts of various partition functions introduced above. It is found that all the (twisted) partition functions $Z^{(M)}_{\mathbf{reg}}$, $\widetilde{Z}^{(M)}_{\mathbf{reg}}$ can be decomposed into the forms like (2.25), and their discrete parts are uniquely determined as sesquilinear forms of the modular completion $\widehat{\chi}_{\mathbf{dis}}$ (C.25). Relevant analyses are quite reminiscent of those given in [1] and we shall not detail them here.

For notational simplicity, we here introduce a new symbol of the modular completion;

$$
\begin{aligned}
&\widehat{\boldsymbol{\chi}}(v,m;\tau,z) \equiv \widehat{\chi}_{\mathbf{dis}}(v,a;\tau,z), \quad \text{with } m \equiv v + 2Ka \in \mathbb{Z}_{2NK}, \quad v = 0,1\ldots,N-1, \\
&\widehat{\boldsymbol{\chi}}(v,m;\tau,z) \equiv 0, \quad \text{if } m - v \notin 2K\mathbb{Z}.
\end{aligned}
\tag{3.14}
$$

It is explicitly written as

$$
\begin{aligned}
\widehat{\boldsymbol{\chi}}(v, v+2Ka;\tau,z) = \sum_{n\in\mathbb{Z}} \frac{\theta_1(\tau,z)}{i\eta(\tau)^3}\, y^{2K\left(n+\frac{a}{N}\right)} q^{NK\left(n+\frac{a}{N}\right)^2} & \left[\frac{(yq^{Nn+a})^{\frac{v}{N}}}{1-yq^{Nn+a}}\right. \\
-\frac{1}{2}\sum_{r\in\mathbb{Z}} \operatorname{sgn}(v+Nr+0)\, \mathrm{Erfc}\left(\sqrt{\frac{\pi\tau_2}{NK}}\,|v+Nr|\right) & \left. y^{\frac{v}{N}+r}\, q^{(v+Nr)\left(n+\frac{a}{N}\right)}\right],
\end{aligned}
\tag{3.15}
$$

where $\mathrm{Erfc}(x)$ denotes the error function (C.20), which acts as a damping factor enough to make the power series convergent. See Appendix C for more detail. We would also use the symbol '$\widehat{\boldsymbol{\chi}}^{[N,K]}(v,m;\tau,z)$' when clarifying the integer parameters N and K[7].

[7] Note that our definition of the extended characters does not only depend on the level $k \equiv N/K$, but on the choice of N and K, because the sum over spectral flow with the flow momenta $n \in N\mathbb{Z}$ is taken.

With this preparation the discrete part of the cigar partition function (2.24) is written as [1];

$$Z_{\mathbf{dis}}(\tau,z)=e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2}\sum_{v=0}^{N-1}\sum_{m+\tilde{m}\in 2N\mathbb{Z}}\widehat{\boldsymbol{\chi}}(v,m;\tau,z)\,[\widehat{\boldsymbol{\chi}}(v,\tilde{m};\tau,z)]^*\,. \tag{3.16}$$

Making similar manipulations, we reach the discrete partition functions for the orbifolds (3.2) and (3.6) as[8]

$$Z_{\mathbf{dis}}^{(M)}(\tau,z\,|\alpha_1,\alpha_2)\;=\;e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2}\sum_{v=0}^{N-1}\sum_{m+\tilde{m}\equiv 2L\alpha_1\ (\mathrm{mod}\ 2LM)}e^{2\pi i\frac{\alpha_2}{2KM}(m-\tilde{m})}\,\widehat{\boldsymbol{\chi}}(v,m;\tau,z)\,[\widehat{\boldsymbol{\chi}}(v,\tilde{m};\tau,z)]^* \tag{3.17}$$

$$\widetilde{Z}_{\mathbf{dis}}^{(M)}(\tau,z\,|\beta_1,\beta_2)\;=\;e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2}\sum_{v=0}^{N-1}\sum_{\substack{m+\tilde{m}\in 2L\mathbb{Z}\\ m-\tilde{m}\equiv 2K\beta_1\ (\mathrm{mod}\ 2KM)}}e^{2\pi i\frac{\beta_2}{2LM}(m+\tilde{m})}\,\widehat{\boldsymbol{\chi}}(v,m;\tau,z)\,[\widehat{\boldsymbol{\chi}}(v,\tilde{m};\tau,z)]^*\,. \tag{3.18}$$

It turns out that they possess the strict modular covariance;

$$\begin{aligned}
&Z_{\mathbf{dis}}^{(M)}(\tau+1,z\,|\,\alpha_1,\alpha_2)=Z_{\mathbf{dis}}^{(M)}(\tau,z\,|\,\alpha_1,\alpha_1+\alpha_2),\quad Z_{\mathbf{dis}}^{(M)}\left(-\frac{1}{\tau},\frac{z}{\tau}\,\middle|\,\alpha_1,\alpha_2\right)=Z_{\mathbf{dis}}^{(M)}(\tau,z\,|\,\alpha_2,-\alpha_1),\\
&\widetilde{Z}_{\mathbf{dis}}^{(M)}(\tau+1,z\,|\,\beta_1,\beta_2)=\widetilde{Z}_{\mathbf{dis}}^{(M)}(\tau,z\,|\,\beta_1,\beta_1+\beta_2),\quad \widetilde{Z}_{\mathbf{dis}}^{(M)}\left(-\frac{1}{\tau},\frac{z}{\tau}\,\middle|\,\beta_1,\beta_2\right)=\widetilde{Z}_{\mathbf{dis}}^{(M)}(\tau,z\,|\,\beta_2,-\beta_1).
\end{aligned} \tag{3.19}$$

These nice features are expected from the fact that the discrete part is uniquely determined from the regularized partition functions and the regularization parameter ϵ is removed safely. We will later prove these formulas directly from the modular transformation formulas of $\widehat{\boldsymbol{\chi}}(*,*;\tau,z)$ given in (C.29), (C.30).

Several remarks are in order;

1. All the Fourier and $\mathbb{Z}_M$-orbifold (or M-fold cover) relations previously discussed are still preserved after separating the discrete parts. For instance, we obtain

$$\widetilde{Z}_{\mathbf{dis}}^{(M)}(\tau,z\,|\,\beta_1,\beta_2)\;=\;\frac{1}{M}\sum_{\alpha_1,\alpha_2\in\mathbb{Z}_M}e^{2\pi i\frac{1}{M}(\alpha_1\beta_2-\alpha_2\beta_1)}\,Z_{\mathbf{dis}}^{(M)}(\tau,z\,|\,\alpha_1,\alpha_2), \tag{3.20}$$

[8]To derive these formulas of 'character decompositions', it seems easy to first work on $\widetilde{Z}_{\mathbf{reg}}^{(M)}(\tau,z\,|\,\beta_1,\beta_2)$, and then to make use of the Fourier transformation relation (3.4). In Appendix D we will present an explicit calculation following that given in [1].

corresponding to (3.4). This fact results from the uniqueness of the relevant decompositions, and is easy to check by comparing directly (3.17) and (3.18).

2. All the discrete partition functions introduced here are independent of the choice of pair (N, K) as long as $k = N/K$ is fixed. As already mentioned, the same statement is obvious by definition for the regularized partition functions $Z_{\mathbf{reg}}^{(*)}$, and again it results from the uniqueness of decompositions. It is, however, non-trivial to check it directly, because the modular completion $\widehat{\boldsymbol{\chi}}(*,*;\tau,z)$ *does* depend on the choice of pair (N, K), not only on $k \equiv N/K$. In doing it, the next identity is useful;

$$\sum_{j\in\mathbb{Z}_s} \widehat{\boldsymbol{\chi}}^{[sN,sK]}(sv, s\{v+2K(a+Nj)\}\,;\tau,z) = \widehat{\boldsymbol{\chi}}^{[N,K]}(v, v+2Ka;\tau,z) \tag{3.21}$$

where we made it explicit the (N, K) dependence of $\widehat{\boldsymbol{\chi}}(*,*;\tau,z)$. This identity is proven by using the definition of $\widehat{\boldsymbol{\chi}}(*,*;\tau,z)$ (C.25).

We also point out that $Z_{\mathbf{dis}}^{(M)}(\tau, z\,|\,0,0)$ does not depend even on M in the similar manner to $Z_{\mathbf{reg}}^{(M)}(\tau, z\,|\,0,0;\epsilon)$. Namely, $Z_{\mathbf{dis}}^{(M)}(\tau, z\,|\,0,0) = Z_{\mathbf{dis}}(\tau, z)$, holds for an arbitrary $M \in \mathbb{Z}_{>0}$.

3. When observing the structures of discrete partition functions (3.17) and (3.18), one may notice a reminiscence of general modular invariants in the parafermion theory [19]. It gets clearer if focusing on the one for the $\mathbb{Z}_M$-orbifold;

$$\begin{aligned} Z_{\mathbf{dis}}^{(M)}(\tau,z) &\equiv \widetilde{Z}_{\mathbf{dis}}^{(M)}(\tau,z\,|0,0) = e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2} \sum_{v=0}^{N-1} \sum_{\substack{m+\tilde{m}\in 2L\mathbb{Z} \\ m-\tilde{m}\in 2KM\mathbb{Z}}} \widehat{\boldsymbol{\chi}}(v,m;\tau,z)\,[\widehat{\boldsymbol{\chi}}(v,\tilde{m};\tau,z)]^* \\ &= e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2} \sum_{v=0}^{N-1} \sum_{r\in\mathbb{Z}_{2KM},\, s\in\mathbb{Z}_L} \widehat{\boldsymbol{\chi}}(v, Lr+KMs;\tau,z)\,[\widehat{\boldsymbol{\chi}}(v, Lr-KMs;\tau,z)]^* \end{aligned} \tag{3.22}$$

which is indeed modular invariant. Especially, for models with integer levels $k = N$, $(K = 1)$, we find[9]

$$\begin{aligned} &Z_{\mathbf{dis}}(\tau,z)\ [\text{axial } SL(2)_N/U(1)] \longleftrightarrow \text{anti-diagonal modular invariant}, \\ &Z_{\mathbf{dis}}^{(N)}(\tau,z)\ [\text{vector } SL(2)_N/U(1)] \longleftrightarrow \text{diagonal modular invariant}. \end{aligned}$$

Of course, this resemblance is not surprising because such a structure has its origin in the $U(1)$-coset and taking suitable orbifolds. A crucial difference is that there exist infinitely many inequivalent models here, while having at most a finite number of modular invariants

[9] $Z_{\mathbf{dis}}^{(N)}(\tau, z)$ always has a diagonal form for an arbitrary level $k = N/K$. However, $Z_{\mathbf{dis}}(\tau, z)$ gets anti-diagonal only when taking $K = 1$.

in the parafermion theory. Note that one can choose an arbitrarily large integer M by scaling $(N,K) \to (MN, MK)$ if necessary.

3.3 'Continuum Limit'

Now, let us discuss a limit of 'infinite order orbifold', which should be also described by the universal cover of trumpet model by T-duality;

$$['\mathbb{Z}\text{-orbifold' of cigar}] \cong [\text{universal cover of trumpet}].$$

We shall take the large M-limit with keeping $k \equiv N/K$ fixed. Since we require M divides N, we have to make N (and K) to be large simultaneously. Hence, it is simplest to set $M = N$ first. We then take the large N-limit with k fixed. Since $Z^{(N)}_{\mathbf{dis}}(\tau, z)$ is a diagonal modular invariant as mentioned above, we have

$$\lim_{z,\bar{z}\to 0} Z^{(N)}_{\mathbf{dis}}(\tau, z) = N,$$

corresponding to the N Ramond vacua with quantum numbers

$$h = \tilde{h} = \frac{\hat{c}}{8}, \quad Q = \widetilde{Q} = \frac{v}{N} - \frac{1}{2}, \quad (v = 0, 1, \ldots, N-1).$$

Hence, the naive large N-limit apparently diverges due to more and more dense distribution of Ramond vacua. We rather define the 'continuum limit' by

$$Z^{(\infty)}_{\mathbf{dis}}(\tau, z) := \lim_{\substack{N\to\infty \\ k\equiv N/K \text{ fixed}}} \frac{1}{N} Z^{(N)}_{\mathbf{dis}}(\tau, z), \tag{3.23}$$

so that it is contributed from the continuum of Ramond vacua located on the segment $-\frac{1}{2} \leq Q \leq \frac{1}{2}$ with unit density. Since the extended characters are defined to be the spectral flow orbits with flow momenta $n \in N\mathbb{Z}$, taking the large N-limit should deconstruct the orbits, leaving the irreducible characters. An explicit calculation yields

$$Z^{(\infty)}_{\mathbf{dis}}(\tau, z) \;=\; e^{-2\pi \frac{\hat{c}}{\tau_2} z_2^2} \sum_{r\in\mathbb{Z}} \frac{1}{k} \int_0^k d\lambda\, \widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, r; \tau, z) \left[\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, r; \tau, z)\right]^*, \tag{3.24}$$

where $\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, r; \tau, z)$ denotes the modular completion of *irreducible* discrete character (C.19) defined by

$$\begin{aligned}
\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n; \tau, z) \;&:=\; \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} y^{\frac{2n}{k}} q^{\frac{n^2}{k}} \left[\frac{(yq^n)^{\frac{\lambda}{k}}}{1 - yq^n} - \frac{1}{2} \sum_{\nu\in\lambda+k\mathbb{Z}} \mathrm{sgn}(\nu+0)\, \mathrm{Erfc}\left(\sqrt{\frac{\pi\tau_2}{k}}\, |\nu|\right) (yq^n)^{\frac{\nu}{k}}\right] \\
&\equiv\; \lim_{\substack{N\to\infty \\ k\equiv N/K \text{ fixed}}} \widehat{\boldsymbol{\chi}}^{[N,K]}(v, v+2Kn; \tau, z), \qquad (0 \leq \lambda \leq k, \quad n \in \mathbb{Z}),
\end{aligned} \tag{3.25}$$

with the identification $v = K\lambda$. The infinite r-sum appearing in (3.24) converges as long as $\tau_2 > 0$, which is shown based on the following facts;

- The irreducible discrete character $\mathrm{ch}_{\mathbf{dis}}(\lambda, n)$ (C.6) (the first term in the first line of (3.25)) obviously well behaves under the large $|n|$-limit.
- By utilizing a simple inequality $\mathrm{Erfc}\left(\sqrt{\frac{\pi\tau_2}{k}}|\nu|\right) \leq e^{-\pi\tau_2\frac{\nu^2}{k}}$, the difference $\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n) - \mathrm{ch}_{\mathbf{dis}}(\lambda, n) =: f_n(\tau, z)\dfrac{\theta_1(z)}{i\eta(\tau)^3}$ is evaluated as

$$|f_n(\tau, z)| \leq C \left|y^{\frac{n}{k}}\right| \, e^{-\pi\tau_2\frac{n^2}{k}},$$

  with some constant C independent of n.

The modular transformation formulas of (3.25) are presented in (C.22), (C.23). It may be amusing that $\widehat{\mathrm{ch}}_{\mathbf{dis}}(*, *; \tau, z)$ includes both continuous and discrete quantum numbers, which are S-transformed in a mixed way.

More generally we define the twisted partition function at this limit by

$$\begin{aligned}\widetilde{Z}^{(\infty)}_{\mathbf{dis}}(\tau, z \,|\, n_1, n_2) \;&:=\; \lim_{\substack{N\to\infty \\ k\equiv N/K \text{ fixed}}} \frac{1}{N}\, \widetilde{Z}^{(N)}_{\mathbf{dis}}(\tau, z \,|\, n_1, n_2) \\ &=\; e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2} \sum_{r\in\mathbb{Z}} \frac{1}{k}\int_0^k d\lambda\, e^{2\pi i\frac{n_2}{k}(\lambda+n_1+2r)}\, \widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n_1+r; \tau, z) \left[\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, r; \tau, z)\right]^*, \\ &\qquad (n_1, n_2 \in \mathbb{Z}). \end{aligned} \tag{3.26}$$

It satisfies the modular covariance relation

$$\widetilde{Z}^{(\infty)}_{\mathbf{dis}}(\tau+1, z \,|\, n_1, n_2) = \widetilde{Z}^{(\infty)}_{\mathbf{dis}}(\tau, z \,|\, n_1, n_1+n_2), \quad \widetilde{Z}^{(\infty)}_{\mathbf{dis}}\left(-\frac{1}{\tau}, \frac{z}{\tau} \,\middle|\, n_1, n_2\right) = \widetilde{Z}^{(\infty)}_{\mathbf{dis}}(\tau, z \,|\, n_2, -n_1), \tag{3.27}$$

which is obvious by construction.

We add a few remarks:

1. Although we took the above limit by setting $M = N$ first, the definition (3.26) does not depend on this procedure. Namely, one can confirm that

$$\widetilde{Z}^{(\infty)}_{\mathbf{dis}}(\tau, z \,|\, n_1, n_2) \;=\; \lim_{\substack{M\to\infty,\ M \text{ divides } N, \\ k\equiv N/K \text{ fixed}}} \frac{1}{M}\, \widetilde{Z}^{(M)}_{\mathbf{dis}}(\tau, z \,|\, n_1, n_2) \tag{3.28}$$

holds. In fact, one can reconstruct the discrete partition function (3.18) from (3.26) as follows ($\beta_1, \beta_2 \in \mathbb{Z}_M$);

$$\widetilde{Z}^{(M)}_{\mathbf{dis}}(\tau, z \,|\, \beta_1, \beta_2) = M \sum_{m_1, m_2 \in \mathbb{Z}} \widetilde{Z}^{(\infty)}_{\mathbf{dis}}(\tau, z \,|\, \beta_1 + Mm_1, \beta_2 + Mm_2), \tag{3.29}$$

and (3.28) is readily obtained from this fact.

2. The modular completion of irreducible character (3.25) is well-defined for general level $k \in \mathbb{R}_{>0}$ and depends continuously on it. Therefore, even though assumed the rational k to derive the above results, one may regard the formulas (3.24), (3.26) as correct ones for general k. However, the extended discrete characters and their modular completion are well-defined only for the rational levels.

3. In deriving (3.24), (3.26), we first integrated the modulus $u \equiv s_1\tau + s_2$ out, and then took the large N-limit. If we instead take the large N-limit first, we would obtain

$$\begin{aligned} Z^{(\infty)}_{\mathbf{reg}}(\tau, z; \epsilon) \;&=\; k e^{-\frac{2\pi}{\tau_2}\frac{k+4}{k} z_2^2} \sum_{m_1, m_2 \in \mathbb{Z}} \int_0^1 dt_1 \int_0^1 dt_2 \int_{\Sigma(z;\epsilon)} \frac{d^2 u}{\tau_2} \\ &\times \left| \frac{\theta_1\left(\tau, u + \frac{k+2}{k} z\right)}{\theta_1\left(\tau, u + \frac{2}{k} z\right)} \right|^2 e^{-4\pi \frac{u_2 z_2}{\tau_2}} \, e^{-\frac{\pi k}{\tau_2} |u + (t_1 + m_1)\tau + (t_2 + m_2)|^2}, \end{aligned} \tag{3.30}$$

as the continuum limit of the regularized partition function. Each twisted sector of the orbifold is parameterized by the continuous parameters t_1, t_2. Note here that the order of integrations is crucial. One has to first integrate the modulus parameter u to achieve the above result (3.24). If inverting the order, we would not gain sensible discrete partition functions with expected modular properties. This subtlety comes from the presence/absence of Gaussian factor in the u-integral, which is necessary to utilize some contour deformation techniques that plays an important role to extract the discrete part. (See [1] and Appendix D.) If making first the t-integration (combined with the summation over m_i), the Gaussian factor drops off, and one could not extract suitable discrete parts.

3.4 Direct Proof of Modular Covariance

As promised before, let us here present a direct proof of the modular covariance of discrete partition functions (3.17), (3.18), and (3.26), that is, the formulas (3.19) and (3.27) based on the modular transformation formulas of $\widehat{\boldsymbol{\chi}}(*, *; \tau, z)$. We only focus on $\widetilde{Z}^{(M)}_{\mathbf{dis}}(\tau, z \,|\, \alpha, \beta)$ (3.18).

After that, remaining formulas for the other partition functions (3.17) and (3.26) can be readily derived by using the Fourier relation (3.20) and the definition (3.26) itself.

For our purpose it is more convenient to rewrite (3.18) in terms of the original notation of modular completion $\widehat{\chi}_{\mathbf{dis}}(v,a;\tau,z)$, in which the discrete parameters v and a are unconstrained;

$$\widetilde{Z}_{\mathbf{dis}}(\tau,z\,|\,\alpha,\beta)=e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2}\sum_{(v,a,\tilde{a})\in\mathcal{R}(M,\alpha)}e^{2\pi i\frac{\beta}{N}\{v+K(a+\tilde{a})\}}\,\widehat{\chi}_{\mathbf{dis}}(v,a;\tau,z)\,[\widehat{\chi}_{\mathbf{dis}}(v,\tilde{a};\tau,z)]^*\,. \quad (3.31)$$

Here the range of summation is defined by

$$\begin{aligned}\mathcal{R}(M,\alpha)\;:=\;&\{(v,a,\tilde{a})\in\mathbb{Z}\times\mathbb{Z}_N\times\mathbb{Z}_N\;;\;0\le v\le N-1,\\&\qquad a-\tilde{a}\equiv\alpha\;(\bmod\,M),\;v+K(a+\tilde{a})\in L\mathbb{Z}\}\,.\end{aligned}\quad (3.32)$$

Consider first the T-transformation. Due to the formula (C.30) we obtain

$$\begin{aligned}\widetilde{Z}^{(M)}_{\mathbf{dis}}(\tau+1,z\,|\,\alpha,\beta)&=e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2}\sum_{(v,a,\tilde{a})\in\mathcal{R}(M,\alpha)}e^{2\pi i\frac{\beta}{N}\{v+K(a+\tilde{a})\}}\,e^{2\pi i\left\{\frac{K}{N}(a^2-\tilde{a}^2)+\frac{v}{N}(a-\tilde{a})\right\}}\\&\qquad\times\widehat{\chi}_{\mathbf{dis}}(v,a;\tau,z)\;[\widehat{\chi}_{\mathbf{dis}}(v,\tilde{a};\tau,z)]^*\\&=e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2}\sum_{(v,a,\tilde{a})\in\mathcal{R}(M,\alpha)}e^{2\pi i\frac{\alpha+\beta}{N}\{v+K(a+\tilde{a})\}}\,\widehat{\chi}_{\mathbf{dis}}(v,a;\tau,z)\;[\widehat{\chi}_{\mathbf{dis}}(v,\tilde{a};\tau,z)]^*\\&=\widetilde{Z}^{(M)}_{\mathbf{dis}}(\tau,z\,|\,\alpha,\alpha+\beta).\end{aligned}\quad (3.33)$$

On the other hand, the calculation of S-transformation is much more complicated. We first obtain from (C.29)

$$\begin{aligned}\widetilde{Z}^{(M)}_{\mathbf{dis}}\left(-\frac{1}{\tau},\frac{z}{\tau}\,|\,\alpha,\beta\right)&=e^{-2\pi\hat{c}\frac{z_2^2}{\tau_2}}\sum_{(v,a,\tilde{a})\in\mathcal{R}(M,\alpha)}e^{2\pi i\frac{\beta}{N}\{v+K(a+\tilde{a})\}}\\&\qquad\times\sum_{v'_L,v'_R=0}^{N-1}\sum_{a'_L,a'_R\in\mathbb{Z}_N}\frac{1}{N}e^{-2\pi i\frac{va'_L+v'_La+2Kaa'_L}{N}}\,\frac{1}{N}e^{2\pi i\frac{va'_R+v'_R\tilde{a}+2K\tilde{a}a'_R}{N}}\\&\qquad\qquad\times\widehat{\chi}_{\mathbf{dis}}(v'_L,a'_L;\tau,z)\;[\widehat{\chi}_{\mathbf{dis}}(v'_R,a'_R;\tau,z)]^*\\&=e^{-2\pi\hat{c}\frac{z_2^2}{\tau_2}}\sum_{\widehat{v}\in L\mathbb{Z}_M}\;\sum_{a,\tilde{a}\in\mathbb{Z}_N,\,a-\tilde{a}\in\alpha+M\mathbb{Z}}\;\sum_{v'_L,v'_R=0}^{N-1}\;\sum_{a'_L,a'_R\in\mathbb{Z}_N}\\&\qquad\times e^{2\pi i\frac{\beta\widehat{v}}{N}}\,\frac{1}{N}e^{-2\pi i\frac{\widehat{v}a'_L+v'_La+2Kaa'_L}{N}}\,\frac{1}{N}e^{2\pi i\frac{\widehat{v}a'_R+v'_R\tilde{a}+2K\tilde{a}a'_R}{N}}\,e^{2\pi i\frac{K}{N}(a+\tilde{a})(a'_L-a'_R)}\\&\qquad\qquad\times\widehat{\chi}_{\mathbf{dis}}(v'_L,a'_L;\tau,z)\;[\widehat{\chi}_{\mathbf{dis}}(v'_R,a'_R;\tau,z)]^*\,.\end{aligned}\quad (3.34)$$

In the second line we set $\widehat{v}:=v+K(a+\tilde{a})$. The $\widehat{v}$-summation is easy to carry out, yielding Kronecker symbol;

$$\delta^{(M)}_{a'_L-a'_R,\beta}\equiv\begin{cases}1 & a'_L-a'_R\equiv\beta\;(\bmod\;M)\\0 & \text{otherwise,}\end{cases}$$

Moreover, rewriting $\tilde{a} = a - \alpha + Mm$, $(m \in \mathbb{Z}_L)$, we obtain

$$
\begin{aligned}
[\text{R.H.S of (3.34)}] \;=\;& e^{-2\pi \hat{c}\frac{z_2^2}{\tau_2}} \sum_{a\in\mathbb{Z}_N} \sum_{m\in\mathbb{Z}_L} \sum_{v'_L, v'_R=0}^{N-1} \sum_{a'_L, a'_R\in\mathbb{Z}_N} \\
& \times M\delta^{(M)}_{a'_L - a'_R, \beta} \frac{1}{N^2} e^{-2\pi i \frac{a}{N}\left(v'_L - v'_L\right)} e^{2\pi i \frac{(-\alpha)}{N}\left\{v'_R + K(a'_L + a'_R)\right\}} e^{2\pi i \frac{m}{L}\left\{v'_R + K(a'_L + a'_R)\right\}} \\
& \times \widehat{\chi}_{\mathbf{dis}}(v'_L, a'_L; \tau, z)\ [\widehat{\chi}_{\mathbf{dis}}(v'_R, a'_R; \tau, z)]^* \\
=\;& e^{-2\pi \hat{c}\frac{z_2^2}{\tau_2}} \sum_{v'_L, v'_R=0}^{N-1} \sum_{a'_L, a'_R\in\mathbb{Z}_N} \delta^{(M)}_{a'_L - a'_R, \beta}\, \delta^{(N)}_{v'_L, v'_R}\, \delta^{(L)}_{v'_R + K(a'_L + a'_R), 0} \\
& \times e^{2\pi i \frac{(-\alpha)}{N}\left\{v'_R + K(a'_L + a'_R)\right\}} \widehat{\chi}_{\mathbf{dis}}(v'_L, a'_L; \tau, z)\ [\widehat{\chi}_{\mathbf{dis}}(v'_R, a'_R; \tau, z)]^* \\
=\;& e^{-2\pi \hat{c}\frac{z_2^2}{\tau_2}} \sum_{(v', a'_L, a'_R)\in\mathcal{R}(M,\beta)} e^{2\pi i \frac{(-\alpha)}{N}\left\{v'_R + K(a'_L + a'_R)\right\}} \widehat{\chi}_{\mathbf{dis}}(v', a'_L; \tau, z)\ [\widehat{\chi}_{\mathbf{dis}}(v', a'_R; \tau, z)]^* \\
\equiv\;& \widetilde{Z}^{(M)}_{\mathbf{dis}}(\tau, z \,|\, \beta, -\alpha).
\end{aligned}
\tag{3.35}
$$

This is the desired result.

4 Elliptic Genera

The elliptic genus [26] is a nice tool to examine important aspects of any $\mathcal{N} = 2$ superconformal filed theory. It is defined by formally setting $\bar{z} = 0$ in the partition function of $\widetilde{\text{R}}$-sector, while leaving z at a generic value. In [1] we studied the elliptic genera of the cigar $SL(2,\mathbb{R})/U(1)$ theory (2.22) and some orbifolds of it. These analyses were based on the character decomposition of partition function mentioned above and also the direct evaluation of path-integration. For instance, the elliptic genus of the simplest cigar model is written (in the notation adopted here) as

$$
\begin{aligned}
\mathcal{Z}(\tau, z) \;=\;& \lim_{\epsilon \to +0} k\, e^{\frac{\pi}{k\tau_2}z^2} \sum_{m_1, m_2\in\mathbb{Z}} \int_{\Sigma(z,\epsilon)} \frac{d^2u}{\tau_2} \frac{\theta_1\left(\tau, u + \frac{k+2}{k}z\right)}{\theta_1\left(\tau, u + \frac{2}{k}z\right)} \\
& \times e^{2\pi i \frac{u_2}{\tau_2}z}\, e^{-\frac{\pi k}{\tau_2}|u + m_1\tau + m_2|^2} \\
=\;& \sum_{v=0}^{N-1} \sum_{\substack{a\in\mathbb{Z}_N \\ v+Ka\in N\mathbb{Z}}} \widehat{\boldsymbol{\chi}}(v, v + 2Ka\,;\, \tau, z).
\end{aligned}
\tag{4.1}
$$

The first line is derived by formally setting $\bar{z} = 0$ in the regularized partition function (2.24). Since the integrand of (4.1) has at most simple poles, the $\epsilon \to +0$ limit converges, and it is easy

to confirm that the first line shows an expected modular behavior;

$$\mathcal{Z}(\tau+1,z)=\mathcal{Z}(\tau,z), \qquad \mathcal{Z}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=e^{i\pi\frac{\hat{c}}{\tau}z^2}\,\mathcal{Z}(\tau,z), \tag{4.2}$$

which is characteristic for the Jacobi form of weight 0 and index $\hat{c}/2$. We should, however, emphasize that $\mathcal{Z}(\tau,z)$ is not holomorphic with respect to τ in a sharp contrast with compact superconformal models.

On the other hand, the second line can be derived in two ways;

(i) One may substitutes the formula of Witten Indices (C.33) into the 'character decomposition' (3.16). Then, it is easy to obtain the second line of (4.1). Note that the 'continuous part' in the decomposition (2.25) does not contribute when setting $\bar{z}=0$.

(ii) One can also directly show the equality between the first and the second lines in (4.1) by means of Poisson resummation and some contour deformation techniques. See [1] for the detail. It would be important as a cross check of calculations. Furthermore, this equality implies good modular behaviors of the modular completions $\widehat{\boldsymbol{\chi}}(*,*;\tau,z)$. In fact, it seems easiest to derive the modular S-transformation formula (C.29) based on such an identity for the orbifold cases discussed below.

Now, the main purpose of this section is to make generalizations of this analysis to various orbifold models given in the previous section. Again we assume $k\equiv N/K$ (N, K are not necessarily assumed to be coprime) and let $N=ML$. Explicit analyses are almost parallel. First of all, corresponding to (3.2), we shall introduce the elliptic genus with the $\mathbb{Z}_M$-twisting as

$$\begin{aligned}\mathcal{Z}^{(M)}(\tau,z\,|\alpha_1,\alpha_2)\equiv \lim_{\epsilon\to+0}\,k\,e^{\frac{\pi}{k\tau_2}z^2}\sum_{m_1,m_2\in\mathbb{Z}}\int_{\Sigma(z,\epsilon)}\frac{d^2u}{\tau_2}\,\frac{\theta_1\left(\tau,u+\frac{k+2}{k}z\right)}{\theta_1\left(\tau,u+\frac{2}{k}z\right)}\\ \times e^{2\pi i\frac{u_2}{\tau_2}z}\,e^{-\frac{\pi k}{\tau_2}\left|u+\left(m_1+\frac{\alpha_1}{M}\right)\tau+\left(m_2+\frac{\alpha_2}{M}\right)\right|^2}.\end{aligned} \tag{4.3}$$

One can show

$$\mathcal{Z}^{(M)}(\tau,z\,|\,\alpha_1,\alpha_2)=\sum_{v=0}^{N-1}\sum_{\substack{a\in\mathbb{Z}_N\\ v+Ka\equiv L\alpha_1\,(\mathrm{mod}\,N)}}e^{2\pi i\frac{\alpha_2}{M}a}\,\widehat{\boldsymbol{\chi}}(v,v+2Ka\,;\,\tau,z), \tag{4.4}$$

in the same way as (4.1). $\mathcal{Z}^{(M)}(\tau,z\,|\,\alpha_1,\alpha_2)$ shows the modular covariance;

$$\begin{aligned}&\mathcal{Z}^{(M)}(\tau+1,z\,|\,\alpha_1,\alpha_2)=\mathcal{Z}^{(M)}(\tau,z\,|\,\alpha_1,\alpha_1+\alpha_2),\\ &\mathcal{Z}^{(M)}\left(-\frac{1}{\tau},\frac{z}{\tau}\,|\,\alpha_1,\alpha_2\right)=e^{i\pi\frac{\hat{c}}{\tau}z^2}\,\mathcal{Z}^{(M)}(\tau,z\,|\,\alpha_2,-\alpha_1).\end{aligned} \tag{4.5}$$

These relations are straightforwardly proven by using the 'path-integral representation' (4.3). One can also derive it based on the 'character decomposition' (4.4) and the modular transformation formulas (C.29), (C.30) as in (3.33), (3.35).

It is also useful to introduce the 'Fourier transform' of (4.3) as in (3.4);

$$\widetilde{\mathcal{Z}}^{(M)}(\tau, z \,|\, \beta_1, \beta_2) \;=\; \frac{1}{M} \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_M} e^{2\pi i \frac{1}{M}(\alpha_1\beta_2 - \alpha_2\beta_1)}\, \mathcal{Z}^{(M)}(\tau, z \,|\, \alpha_1, \alpha_2), \tag{4.6}$$

which again has the modular covariance. We can explicitly compute it as

$$\begin{aligned}\widetilde{\mathcal{Z}}^{(M)}(\tau, z \,|\beta_1, \beta_2) \;&=\; \lim_{\epsilon \to +0} M\, e^{\frac{\pi}{k\tau_2}z^2} \sum_{n_1, n_2 \in \mathbb{Z}} \int_{\Sigma(z,\epsilon)} \frac{d^2u}{\tau_2} \frac{\theta_1\left(\tau, u + \frac{k+2}{k}z\right)}{\theta_1\left(\tau, u + \frac{2}{k}z\right)} \\ &\qquad \times e^{2\pi i \frac{u_2}{\tau_2} z}\, e^{-\frac{\pi}{k\tau_2}|(Mn_1+\beta_1)\tau + (Mn_2+\beta_2)|^2}\, e^{2\pi i \{(Mn_2+\beta_2)s_1 - (Mn_1+\beta_1)s_2\}} \\ &=\; \sum_{v=0}^{N-1} \sum_{\substack{v+Ka \in L\mathbb{Z} \\ a \equiv \beta_1 \, (\mathrm{mod}\, M)}} e^{2\pi i \frac{\beta_2}{N}(v+Ka)}\, \widehat{\boldsymbol{\chi}}(v, v+2Ka\,;\,\tau, z).\end{aligned} \tag{4.7}$$

It is obvious that

$$\mathcal{Z}^{(M)}(\tau, z \,|\, 0, 0) = \mathcal{Z}(\tau, z), \quad (\text{for } {}^{\forall}M) \tag{4.8}$$

by definition, while we have

$$\widetilde{\mathcal{Z}}^{(M)}(\tau, z \,|\, 0, 0) = \sum_{v=0}^{N-1} \sum_{\substack{v+Ka \in L\mathbb{Z} \\ a \in M\mathbb{Z}}} \widehat{\boldsymbol{\chi}}(v, v+2Ka\,;\,\tau, z), \tag{4.9}$$

which is identified with the elliptic genus of $\mathbb{Z}_M$-orbifold of cigar;

$$\mathcal{Z}^{(M)}(\tau, z) \equiv \frac{1}{M} \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_M} \mathcal{Z}^{(M)}(\tau, z \,|\, \alpha_1, \alpha_2). \tag{4.10}$$

Especially, in the special case $M = N$, we obtain

$$\widetilde{\mathcal{Z}}^{(N)}(\tau, z \,|\, \beta_1, \beta_2) = \sum_{v=0}^{N-1} e^{2\pi i \frac{\beta_2}{N}(v+K\beta_1)}\, \widehat{\boldsymbol{\chi}}(v, v+2K\beta_1\,;\,\tau, z), \tag{4.11}$$

and

$$\mathcal{Z}^{(N)}(\tau, z) \equiv \widetilde{\mathcal{Z}}^{(N)}(\tau, z \,|\, 0, 0) = \sum_{v=0}^{N-1} \widehat{\boldsymbol{\chi}}(v, v\,;\,\tau, z), \tag{4.12}$$

which has been already derived in [3, 1], and nicely expressible in terms of the modular completion of the higher level Appell function [27] given in [15].

Finally, for the 'continuum limit' describing the universal cover of trumpet, we achieve the next formula;

$$\begin{aligned}\widetilde{\mathcal{Z}}^{(\infty)}(\tau, z \,|\, n_1, n_2) &:= \lim_{\substack{M\to\infty,\, M \text{ divides } N \\ k\equiv N/K \text{ fixed}}} \frac{1}{M}\, \widetilde{\mathcal{Z}}^{(M)}(\tau, z \,|\, n_1, n_2) \\ &= \frac{1}{k}\int_0^k d\lambda\, e^{2\pi i \frac{n_2}{k}(\lambda+n_1)}\, \widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n_1; \tau, z), \quad (n_1,\, n_2 \in \mathbb{Z}). \end{aligned} \tag{4.13}$$

The formula (4.13) reads as the Fourier transform of the modular completion of irreducible discrete character $\widehat{\mathrm{ch}}_{\mathbf{dis}}(*, *; \tau, z)$. It is worthwhile to point out that the modular covariance of $\widetilde{\mathcal{Z}}^{(\infty)}(\tau, z \,|\, n_1, n_2)$ (in the same form as (4.5)) is equivalent with the modular transformation formulas of $\widehat{\mathrm{ch}}_{\mathbf{dis}}(*, *; \tau, z)$ (C.22), (C.23). In section 5, we will make use of (4.13) as fundamental building blocks to construct general modular invariants.

We remark that the order of limiting procedure is again important. In deriving (4.13), we have to first make the integration of modulus u, and then should take the large M-limit. If inverting the order, the Gaussian factor for the u-integration drops off, and the resultant elliptic genus does not coincide with (4.13). It does not possess expected properties of elliptic genus contrary to (4.13)[10]. In other words, the 'path-integral representation' of elliptic genera (such as the first line of (4.1)) would be unambiguously defined only for finite order orbifolds.

5 Classification of General Modular Invariants

We finally discuss a classification of general modular invariants. In this section we consider models with arbitrary $k > 0$, which may be irrational. To be more specific, the questions we would like to ask are as follows;

(1) What is the most general candidate of elliptic genus possessing the suitable modular property (as a Jacobi form), when assuming the expression;

$$\mathcal{Z}(\tau, z) = \frac{1}{k}\int_0^k d\lambda \sum_{n\in\mathbb{Z}} \rho(\lambda, n)\, \widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n; \tau, z), \tag{5.1}$$

with some density function $\rho(\lambda, n)$?

[10] More precisely speaking, we would lose the holomorphicity with respect to the angle variable z despite the original definition of elliptic genus, even though the correct modular properties are maintained.

(2) What is the most general candidate of 'discrete partition function' that is modular invariant and written in the form as

$$Z_{\mathbf{dis}}(\tau,z) = e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2}\,\frac{1}{k^2}\int_0^k d\lambda_L \int_0^k d\lambda_R \sum_{n_L,n_R\in\mathbb{Z}} \times\sigma(\lambda_L,\lambda_R,n_L,n_R)\,\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda_L,n_L;\tau,z)\,\left[\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda_R,n_R;\tau,z)\right]^* \qquad (5.2)$$

with some density function $\sigma(\lambda_L,\lambda_R,n_L,n_R)$?

At first glance, it would not appear so easy to solve these classification problems. However, the problems get easier if we rephrase them by using (4.13) as the fundamental building blocks. Note that (4.13) is just the Fourier transform of $\widehat{\mathrm{ch}}_{\mathbf{dis}}$ and the generality of assumption is maintained. For notational simplicity we introduce

$$\Psi_{\boldsymbol{n}}(\tau,z) := \widetilde{\mathcal{Z}}^{(\infty)}(\tau,z\,|\,n_1,n_2), \qquad (\boldsymbol{n}\equiv(n_1,n_2)\in\mathbb{Z}^2), \qquad (5.3)$$

where the R.H.S is given in (4.13). The modular property of the building block $\Psi_{\boldsymbol{n}}$ is very simple;

$$\Psi_{\boldsymbol{n}}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) = e^{i\pi\hat{c}\frac{cz^2}{c\tau+d}}\,\Psi_{\boldsymbol{n}A}(\tau,z), \qquad {}^\forall A\equiv\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z}) \qquad (5.4)$$

It is also convenient to introduce the 'spectral flow operator' $s_{\boldsymbol{n}}$ (${}^\forall\boldsymbol{n}\equiv(n_1,n_2)\in\mathbb{Z}^2$) defined by

$$s_{\boldsymbol{n}}\cdot f(\tau,z) = (-1)^{n_1+n_2}e^{2\pi i\frac{n_1n_2}{k}}q^{\frac{\hat{c}}{2}n_1^2}y^{\hat{c}n_1}\,f(\tau,z+n_1\tau+n_2), \qquad (5.5)$$

for any function $f(\tau,z)$. Then, $\Psi_{\boldsymbol{n}}$ can be expressed as

$$\Psi_{\boldsymbol{n}}(\tau,z) = s_{\boldsymbol{n}}\cdot\Psi_{\boldsymbol{0}}(\tau,z) \equiv s_{\boldsymbol{n}}\cdot\frac{1}{k}\int_0^k d\lambda\,\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda,0;\tau,z). \qquad (5.6)$$

Moreover, because of the useful relation;

$$s_{\boldsymbol{m}}\circ s_{\boldsymbol{n}} = e^{-2\pi i\frac{1}{k}\langle\boldsymbol{m},\boldsymbol{n}\rangle}\,s_{\boldsymbol{m}+\boldsymbol{n}}, \qquad (5.7)$$

where $\langle\ ,\ \rangle$ denotes an $SL(2,\mathbb{Z})$-invariant product;

$$\langle\boldsymbol{m},\boldsymbol{n}\rangle := m_1n_2 - m_2n_1, \qquad (\boldsymbol{m}\equiv(m_1,m_2),\quad \boldsymbol{n}\equiv(n_1,n_2)) \qquad (5.8)$$

we obtain

$$s_{\boldsymbol{m}}\cdot\Psi_{\boldsymbol{n}} = e^{-2\pi i\frac{1}{k}\langle\boldsymbol{m},\boldsymbol{n}\rangle}\,\Psi_{\boldsymbol{m}+\boldsymbol{n}}, \qquad ({}^\forall\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^2). \qquad (5.9)$$

Now, since $\Psi_{\boldsymbol{n}}$ is the Fourier transform of $\widehat{\mathrm{ch}}_{\mathbf{dis}}$, one can rewrite the ansatz (5.1) in the form of

$$\mathcal{Z}^{[a]}(\tau, z) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} a(\boldsymbol{n})\, \Psi_{\boldsymbol{n}}(\tau, z). \tag{5.10}$$

Moreover, by requiring the modular property;

$$\mathcal{Z}^{[a]}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = e^{i\pi\hat{c}\frac{cz^2}{c\tau+d}}\, \mathcal{Z}^{[a]}(\tau, z), \tag{5.11}$$

and using (5.4), one can obtain a simple constraint;

$$a(\boldsymbol{n}A) = a(\boldsymbol{n}), \qquad ({}^{\forall}A \in SL(2,\mathbb{Z})). \tag{5.12}$$

If satisfying it, this is actually the most general function possessing the modular property (5.11) as long as assuming the ansatz (5.1).

Similarly, the most general modular invariant of the form (5.2) is given by

$$Z^{[c]}_{\mathbf{dis}}(\tau, z) = e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2} \sum_{\boldsymbol{n}_L, \boldsymbol{n}_R\in\mathbb{Z}^2} c(\boldsymbol{n}_L, \boldsymbol{n}_R)\, \Psi_{\boldsymbol{n}_L}(\tau, z)\, \left[\Psi_{\boldsymbol{n}_R}(\tau, z)\right]^*, \tag{5.13}$$

where $c(\boldsymbol{n}_L, \boldsymbol{n}_R)$ are arbitrary $SL(2,\mathbb{Z})$-invariant coefficients;

$$c(\boldsymbol{n}_L A,\, \boldsymbol{n}_R A) = c(\boldsymbol{n}_L, \boldsymbol{n}_R), \qquad ({}^{\forall}A \in SL(2,\mathbb{Z})).$$

Let us elaborate on the concrete examples:

1. Elliptic Genera:

(i) $a(\boldsymbol{n}) = \delta_{\boldsymbol{n},\mathbf{0}}$:

This is the simplest example, and (5.10) reduces to the elliptic genus of universal cover of the trumpet;

$$\mathcal{Z}^{[a]}(\tau, z) = \Psi_{\mathbf{0}}(\tau, z) = \mathcal{Z}^{(\infty)}(\tau, z) \equiv \frac{1}{k}\int_0^k d\lambda\, \widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, 0; \tau, z) \tag{5.14}$$

(ii) $a(\boldsymbol{n}) = 1$, $({}^{\forall}\boldsymbol{n} \in \mathbb{Z}^2)$:

In this case, we obtain

$$\mathcal{Z}^{[a]}(\tau, z) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \Psi_{\boldsymbol{n}}(\tau, z). \tag{5.15}$$

In case of $k = N/K$ this is equal the elliptic genus of original cigar model $\mathcal{Z}(\tau, z)$ (4.1). Note that the expression (5.15) is still well-defined even if k is not rational, and one may regard it as the general formula of $\mathcal{Z}(\tau, z)$.

(iii) $a(\boldsymbol{n}) = M\delta^{(M)}_{\boldsymbol{n},\boldsymbol{0}}$:

In this case (5.10) reduces to

$$\mathcal{Z}^{[a]}(\tau,z) = M \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \Psi_{M\boldsymbol{n}}(\tau,z). \tag{5.16}$$

In case of $k = N/K$ this is equal the elliptic genus of $\mathbb{Z}_M$-orbifold of cigar $\widetilde{Z}^{(M)}(\tau,z) \equiv \widetilde{\mathcal{Z}}^{(M)}(\tau,z\,|\,0,0)$ (4.10). More generally, we can prove

$$\widetilde{\mathcal{Z}}^{(M)}(\tau,z|\beta_1,\beta_2) = M \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \Psi_{\boldsymbol{\beta}+M\boldsymbol{n}}(\tau,z), \quad ({}^\forall\boldsymbol{\beta} \equiv (\beta_1,\beta_2) \in \mathbb{Z}_M), \tag{5.17}$$

which is an analogue of the identity (3.29). One may again regard it as a formula applicable to general k, not necessarily rational.

2. Discrete Partition Functions:

(i) $c(\boldsymbol{n}_L,\boldsymbol{n}_R) = \delta_{\boldsymbol{n}_L,\boldsymbol{n}_R}$:

This has the diagonal form, and (5.13) corresponds to the universal cover of trumpet (3.24);

$$Z^{[c]}_{\mathbf{dis}}(\tau,z) = e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2} \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \Psi_{\boldsymbol{n}}(\tau,z)\,[\Psi_{\boldsymbol{n}}(\tau,z)]^* = Z^{(\infty)}_{\mathbf{dis}}(\tau,z), \tag{5.18}$$

More generally the twisted partition function (3.26) is rewritten as ($\boldsymbol{n} \equiv (n_1,n_2) \in \mathbb{Z}^2$)

$$\begin{aligned} Z^{(\infty)}_{\mathbf{dis}}(\tau,z\,|\,n_1,n_2) &= e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} s_{\boldsymbol{n}}\cdot\Psi_{\boldsymbol{m}}(\tau,z)\,[\Psi_{\boldsymbol{m}}(\tau,z)]^* \\ &= e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} e^{-2\pi i\frac{1}{k}\langle\boldsymbol{n},\boldsymbol{m}\rangle}\,\Psi_{\boldsymbol{m}+\boldsymbol{n}}(\tau,z)\,[\Psi_{\boldsymbol{m}}(\tau,z)]^*. \end{aligned} \tag{5.19}$$

(ii) $c(\boldsymbol{n}_L,\boldsymbol{n}_R) = e^{-2\pi i\frac{1}{k}\langle\boldsymbol{n}_L,\boldsymbol{n}_R\rangle}$:

This coefficient is indeed $SL(2,\mathbb{Z})$-invariant. The corresponding partition function is found to be the one of the original cigar model (3.16);

$$Z^{[c]}_{\mathbf{dis}}(\tau,z) = e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2} \sum_{\boldsymbol{n}_L,\boldsymbol{n}_R\in\mathbb{Z}^2} e^{-2\pi i\frac{1}{k}\langle\boldsymbol{n}_L,\boldsymbol{n}_R\rangle}\,\Psi_{\boldsymbol{n}_L}(\tau,z)\,[\Psi_{\boldsymbol{n}_R}(\tau,z)]^* = Z_{\mathbf{dis}}(\tau,z), \tag{5.20}$$

Again this formula works for general $k \in \mathbb{R}_{>0}$.

(iii) $c(\boldsymbol{n}_L, \boldsymbol{n}_R) = M e^{-2\pi i \frac{1}{k}\langle \boldsymbol{n}_L, \boldsymbol{n}_R\rangle} \delta^{(M)}_{\boldsymbol{n}_L, \boldsymbol{n}_R}$:

In this case (5.10) corresponds to the $\mathbb{Z}_M$-orbifold;

$$Z^{[c]}_{\mathbf{dis}}(\tau, z) = e^{-2\pi \frac{\hat{c}}{\tau_2} z_2^2}\, M \sum_{\substack{\boldsymbol{n}_L, \boldsymbol{n}_R \in \mathbb{Z}^2 \\ \boldsymbol{n}_L - \boldsymbol{n}_R \in M\mathbb{Z}^2}} e^{-2\pi i \frac{1}{k}\langle \boldsymbol{n}_L, \boldsymbol{n}_R\rangle}\, \Psi_{\boldsymbol{n}_L}(\tau, z)\, [\Psi_{\boldsymbol{n}_R}(\tau, z)]^* = Z^{(M)}_{\mathbf{dis}}(\tau, z), \tag{5.21}$$

In fact, one can show that it is equal to (3.22) in case of $k = N/K$.

More generally, we can prove

$$\widetilde{Z}^{(M)}_{\mathbf{dis}}(\tau, z \,|\, \beta_1, \beta_2) \;=\; e^{-2\pi \frac{\hat{c}}{\tau_2} z_2^2}\, M \sum_{\substack{\boldsymbol{n}_L, \boldsymbol{n}_R \in \mathbb{Z}^2 \\ \boldsymbol{n}_L - \boldsymbol{n}_R \in \boldsymbol{\beta} + M\mathbb{Z}^2}} e^{-2\pi i \frac{1}{k}\langle \boldsymbol{n}_L, \boldsymbol{n}_R\rangle}\, \Psi_{\boldsymbol{n}_L}(\tau, z)\, [\Psi_{\boldsymbol{n}_R}(\tau, z)]^*, \qquad ({}^{\forall}\boldsymbol{\beta} \equiv (\beta_1, \beta_2) \in \mathbb{Z}_M), \tag{5.22}$$

which is essentially the identity equivalent with (3.29).

6 Summary and Discussions

We first summarize new results given in this paper that extend the work [1];

- We have studied the $\mathbb{Z}_M$-orbifolds of the cigar SUSY $SL(2,\mathbb{R})/U(1)$-coset with a rational level $k = N/K$ and the M-fold covers of trumpet $SL(2,\mathbb{R})/U(1)$-coset with an arbitrary integer M, which are related by the T-duality relations. We have extracted contributions of the BPS representations ('discrete characters') in such a way that good modular properties are preserved. The modular completions of the extended discrete characters introduced in [1] work as suitable building blocks in every model of orbifold or covering space. Especially, the elliptic genera are naturally expressed in terms only of these modular completions in all the models.

- We have further examined a large M-limit (the 'continuum limit'). Both the discrete part of partition function and the elliptic genus have been expanded by the modular completions of *irreducible* discrete characters. In a sharp contrast with any RCFT, we have an infinite number of building blocks at this limit, including both discrete and continuous quantum numbers *being S-transformed in a mixed way.* It would not be likely that modular invariants of such a kind have been known until now. This limit is geometrically identified with the universal cover of trumpet model.

- We also discussed cases of general level k, *allowed to be non-rational.* The discrete part is still well-defined, and is well captured by the modular completions of irreducible characters. General modular invariants for the discrete part are classified for an arbitrary k, in which the Fourier transforms of irreducible modular completions (5.3) play a crucial role. In the cases of rational k, general modular invariants have forms mimicking the parafermion theory. However, there exist an infinite number of inequivalent classes of modular invariants as opposed to the parafermion case. We also note that the solution (5.13) includes much broader class of modular invariants. For instance, non-diagonal modular invariants with $\lambda_L \neq \lambda_R$ are possible, although it is not yet obvious whether they are physically interpretable in the context of 2D black-hole models.

Probably, one of important issues we should discuss is the origin of non-holomorphicity of elliptic genus, or equivalently, an apparent lack of holomorphic factorization in the relevant models. If respecting simple modular properties, which is our stand point in this paper, the modular completions should be fundamental building blocks. On the other hand, if respecting the holomorphic factorization, the partition functions are expanded by an infinite number of extended or irreducible continuous (non-BPS) characters *which may be non-unitary*, in addition to the discrete (BPS) ones. In the latter picture, the emergence of non-unitary characters would cause an IR-instability. However, summing up infinite characters and making an analytic continuation, we can gain an IR-stable partition function, as is consistent with the former picture. This power series would promote extra singularities, which cancel the zeros of θ_1-factors and contribute to the Witten index. This is indeed the possible origin of non-holomorphicity of the elliptic genus. More detailed study on this issue should be one of our future works.

Such an incompatibility between the simple modular behavior and the holomorphic factorization seems to be a characteristic feature of non-compact conformal field theories that include both discrete and continuous spectra of normalizable states. Searching other models showing similar aspects is surely an interesting subject.

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Appendix A: Conventions for Theta Functions

We assume $\tau \equiv \tau_1 + i\tau_2$, $\tau_2 > 0$ and set $q := e^{2\pi i\tau}$, $y := e^{2\pi iz}$;

$$\begin{aligned}
\theta_1(\tau,z) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\sin(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m),\\
\theta_2(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\cos(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m),\\
\theta_3(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2}y^n \equiv \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}),\\
\theta_4(\tau,z) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}y^n \equiv \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2}).
\end{aligned} \tag{A.1}$$

$$\Theta_{m,k}(\tau,z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2}y^{k(n+\frac{m}{2k})}. \tag{A.2}$$

We also set

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{A.3}$$

The spectral flow properties of theta functions are summarized as follows;

$$\begin{aligned}
&\theta_1(\tau, z+m\tau+n) = (-1)^{m+n}q^{-\frac{m^2}{2}}y^{-m}\theta_1(\tau,z)\ ,\\
&\theta_2(\tau, z+m\tau+n) = (-1)^{n}q^{-\frac{m^2}{2}}y^{-m}\theta_2(\tau,z)\ ,\\
&\theta_3(\tau, z+m\tau+n) = q^{-\frac{m^2}{2}}y^{-m}\theta_3(\tau,z)\ ,\\
&\theta_4(\tau, z+m\tau+n) = (-1)^{m}q^{-\frac{m^2}{2}}y^{-m}\theta_4(\tau,z)\ ,\\
&\Theta_{a,k}(\tau, 2(z+m\tau+n)) = q^{-km^2}y^{-2km}\Theta_{a+2km,k}(\tau,2z)\ .
\end{aligned} \tag{A.4}$$

Appendix B: Relevant Path-integral Formulas

We summarize several path-integral formulas relevant to our analysis.

H^3_+**-sector:** [28]

$$Z_g^{(V)}(\tau,u) \equiv \int \mathcal{D}g\, \exp\left[-\kappa S^{(V)}(g,\, h^u,\, h^{u\,\dagger})\right] = \frac{e^{2\pi \frac{u_2^2}{\tau_2}}}{\sqrt{\tau_2}|\theta_1(\tau,u)|^2}, \tag{B.1}$$

$$Z_g^{(A)}(\tau,z) \equiv \int \mathcal{D}g\, \exp\left[-\kappa S^{(A)}(g,\, h^u,\, h^{u\,\dagger})\right] = \frac{e^{2\pi \frac{u_2^2}{\tau_2} - \pi\kappa \frac{|u|^2}{\tau_2}}}{\sqrt{\tau_2}|\theta_1(\tau,u)|^2}. \tag{B.2}$$

$U(1)$**-sector (**Y**-sector):**

$$\begin{aligned} Z_Y^{(A)}(\tau,u) &\equiv \int \mathcal{D}Y\, \exp\left[-\frac{1}{\pi\alpha'}\int d^2w\, |\partial_{\bar{w}} Y^u|^2\right] \\ &= \frac{\sqrt{k}}{\sqrt{\tau_2}\,|\eta(\tau)|^2} \sum_{m_1,m_2\in\mathbb{Z}} e^{-\frac{\pi k}{\tau_2}|(m_1+s_1)\tau+(m_2+s_2)|^2} \end{aligned} \tag{B.3}$$

$$\begin{aligned} Z_Y^{(V)}(\tau,u) &\equiv \int \mathcal{D}Y\, \exp\left[-\frac{1}{\pi\alpha'}\int d^2w\, |\partial_{\bar{w}} Y|^2 - \frac{ik}{2\pi}\int_\Sigma d\Phi[u]\wedge dY\right] \\ &= \frac{\sqrt{k}}{\sqrt{\tau_2}\,|\eta(\tau)|^2} \sum_{\substack{n_1,n_2\in\mathbb{Z} \\ kn_i\in\mathbb{Z}}} e^{-\frac{\pi k}{\tau_2}|n_1\tau+n_2|^2}\, e^{-2\pi i k(s_1 n_2 - s_2 n_1)}. \end{aligned} \tag{B.4}$$

In the axial case (B.3), we set $Y^u \equiv Y + \Phi[u]$, which satisfies the twisted boundary condition (2.21). In the vector case (B.4), we assume k to be rational.

fermion sector:

$$Z_\psi(\tau,u) \equiv \int \mathcal{D}[\psi^\pm,\tilde{\psi}^\pm]\, \exp\left[-S_\psi^{(A)}\left(\psi^\pm,\tilde{\psi}^\pm,a[u]\right)\right] = e^{-2\pi\frac{u_2^2}{\tau_2}} \left|\frac{\theta_1(\tau,u)}{\eta(\tau)}\right|^2. \tag{B.5}$$

ghost sector:

$$Z_{\rm gh}(\tau) \equiv \int \mathcal{D}[b,\tilde{b},c,\tilde{c}]\, \exp\left[-S_{\rm gh}(b,\tilde{b},c,\tilde{c})\right] = \tau_2\,|\eta(\tau)|^4. \tag{B.6}$$

Appendix C: Irreducible and Extended Characters and their Modular Completions

In this appendix we summarize the definitions as well as useful formulas for the (extended) characters and their modular completions of the $\mathcal{N}=2$ superconformal algebra with $\hat{c}\left(\equiv \frac{c}{3}\right) =$

$1+\frac{2}{k}$. We focus only on the $\widetilde{\text{R}}$-sector[11], and when treating the extended characters, we assume $k = N/K$, $(N, K \in \mathbb{Z}_{>0})$ (but, not assume N and K are co-prime).

Continuous (non-BPS) Characters:

$$\mathrm{ch}(P,\mu;\tau,z) := q^{\frac{P^2+\mu^2}{4k}} y^{\frac{\mu}{k}} \frac{\theta_1(\tau,z)}{i\eta(\tau)^3}, \tag{C.1}$$

which is associated to the irrep. with the following conformal weight h and $U(1)$-charge Q;

$$h = \frac{P^2+\mu^2}{4k} + \frac{\hat{c}}{8}, \quad Q = \frac{\mu}{k} \pm \frac{1}{2}, \quad \text{(doubly degenerated)} \tag{C.2}$$

The modular transformation formulas and the spectral flow property are summarized as

$$\mathrm{ch}\left(P,\mu;-\frac{1}{\tau},\frac{z}{\tau}\right) = (-i)e^{i\pi\frac{\hat{c}}{\tau}z^2}\,\frac{1}{2k}\int_{-\infty}^{\infty} dP' \int_{-\infty}^{\infty} d\mu'\, e^{2\pi i\frac{PP'-\mu\mu'}{2k}}\,\mathrm{ch}(P',\mu';\tau,z). \tag{C.3}$$

$$\mathrm{ch}\left(P,\mu;\tau+1,z\right) = e^{2\pi i\frac{P^2+\mu^2}{4k}}\,\mathrm{ch}\left(P,\mu;\tau,z\right). \tag{C.4}$$

$$\mathrm{ch}(P,\mu;\tau,z+r\tau+s) = (-1)^{r+s}e^{2\pi i\frac{\mu}{k}s}q^{-\frac{\hat{c}}{2}r^2}y^{-\hat{c}r}\,\mathrm{ch}(P,\mu+2r;\tau,z), \quad ({}^{\forall}r,s\in\mathbb{Z}). \tag{C.5}$$

Discrete Characters: [29, 30]

$$\mathrm{ch}_{\mathbf{dis}}(\lambda,n;\tau,z) := \frac{(yq^n)^{\frac{\lambda}{k}}}{1-yq^n}\,y^{\frac{2n}{k}}q^{\frac{n^2}{k}}\,\frac{\theta_1(\tau,z)}{i\eta(\tau)^3}, \tag{C.6}$$

which is associated to the n-th spectral flow of discrete irrep. generated by the Ramond vacua[12];

$$h = \frac{\hat{c}}{8}, \quad Q = \frac{\lambda}{k} - \frac{1}{2}, \quad (0 \le \lambda \le k) \tag{C.7}$$

---

[11] In this paper we shall use the convention of $\widetilde{\text{R}}$-characters with the inverse sign compared to those of [1, 7, 9], so that the Witten indices appear with the positive sign. (See (C.33) below.)

[12] The unitarity requires $-\frac{\hat{c}}{2} \le Q \le \frac{\hat{c}}{2}$ for the Ramond vacua, which is equivalent with the condition : $-1 \le \lambda \le k+1$ [29]. The quantum number λ is identified with $2j-1$, where j is the 'isospin' of $SL(2,\mathbb{R})$ in the $SL(2,\mathbb{R})/U(1)$-coset [31]. Thus, the unitarity range $-1 \le \lambda \le k+1$ corresponds to the 'analogue of integrable representations' $0 \le j \le \frac{k+2}{2} \equiv \frac{\kappa}{2}$, where κ denotes the level of bosonic $SL(2,\mathbb{R})$-WZW. The range $0 \le \lambda \le k$ ($\Leftrightarrow$ $\frac{1}{2} \le j \le \frac{k+1}{2}$) that we adopt here is strictly narrower than this 'unitarity range'. This restriction has a clear origin in the discrete spectrum of the SUSY $SL(2,\mathbb{R})/U(1)$-coset read off from the torus partition function. It is worth pointing out that this range is invariant under modular transformations given below. We also note that the missing 'edge' points $\lambda = -1, k+1$ correspond to the 'graviton representation' and its spectral flows, which obey different character formulas (see [30]). This type restriction of spectrum has been already discussed in [32, 33, 7].

The modular transformation formulas are given as [10]

$$\mathrm{ch}_{\mathbf{dis}}\left(\lambda, n; -\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{i\pi\frac{\hat{c}}{\tau}z^2}\left[\frac{1}{k}\int_0^k d\lambda' \sum_{n'\in\mathbb{Z}} e^{2\pi i\frac{\lambda\lambda'-(\lambda+2n)(\lambda'+2n')}{2k}}\mathrm{ch}_{\mathbf{dis}}(\lambda', n'; \tau, z)\right.$$
$$\left. -\frac{i}{2k}\int_{-\infty}^{\infty} d\mu'\, e^{-2\pi i\frac{(\lambda+2n)\mu'}{2k}}\int_{\mathbb{R}+i0} dP'\,\frac{e^{-2\pi\frac{\lambda P'}{2k}}}{1-e^{-\pi(P'+i\mu')}}\,\mathrm{ch}(P', \mu'; \tau, z)\right], \quad \text{(C.8)}$$

$$\mathrm{ch}_{\mathbf{dis}}(\lambda, n; \tau+1, z) = e^{2\pi i\frac{n}{k}(\lambda+n)}\,\mathrm{ch}_{\mathbf{dis}}(\lambda, n; \tau, z)\,. \quad \text{(C.9)}$$

The spectral flow property is written as

$$\mathrm{ch}_{\mathbf{dis}}(\lambda, n; \tau, z+r\tau+s) = (-1)^{r+s}e^{2\pi i\frac{\lambda+2n}{k}s}q^{-\frac{\hat{c}}{2}r^2}y^{-\hat{c}r}\,\mathrm{ch}_{\mathbf{dis}}(\lambda, n+r; \tau, z), \quad ({}^{\forall}r, s\in\mathbb{Z}). \quad \text{(C.10)}$$

Extended Continuous (non-BPS) Characters [10, 7]:

$$\begin{aligned}\chi_{\mathbf{con}}(p, m; \tau, z) &:= \sum_{n\in N\mathbb{Z}}(-1)^n q^{\frac{\hat{c}}{2}n^2}y^{\hat{c}n}\,\mathrm{ch}\left(\frac{p}{K}, \frac{m}{K}; \tau, z+n\tau\right)\\ &= q^{\frac{p^2}{4NK}}\Theta_{m,NK}\left(\tau, \frac{2z}{N}\right)\frac{\theta_1(\tau, z)}{i\eta(\tau)^3}.\end{aligned} \quad \text{(C.11)}$$

This corresponds to the spectral flow sum of the non-degenerate representation with $h = \frac{p^2+m^2}{4NK} + \frac{\hat{c}}{8}$, $Q = \frac{m}{N} \pm \frac{1}{2}$ ($p \geq 0$, $m \in \mathbb{Z}_{2NK}$), whose flow momenta are taken to be $n \in N\mathbb{Z}$. The modular and spectral flow properties are simply written as

$$\chi_{\mathbf{con}}\left(p, m; -\frac{1}{\tau}, \frac{z}{\tau}\right) = (-i)e^{i\pi\frac{\hat{c}}{\tau}z^2}\frac{1}{2NK}\int_{-\infty}^{\infty}dp' \sum_{m'\in\mathbb{Z}_{2NK}} e^{2\pi i\frac{pp'-mm'}{2NK}}\chi_{\mathbf{con}}(p', m'; \tau, z). \quad \text{(C.12)}$$

$$\chi_{\mathbf{con}}(p, m; \tau+1, z) = e^{2\pi i\frac{p^2+m^2}{4NK}}\chi_{\mathbf{con}}(p, m; \tau, z)\,, \quad \text{(C.13)}$$

$$\chi_{\mathbf{con}}(p, m; \tau, z+r\tau+s) = (-1)^{r+s}e^{2\pi i\frac{m}{N}s}q^{-\frac{\hat{c}}{2}r^2}y^{-\hat{c}r}\chi_{\mathbf{con}}(p, m+2Kr; \tau, z), \quad ({}^{\forall}r, s\in\mathbb{Z}). \quad \text{(C.14)}$$

Extended Discrete (BPS) Characters [10, 7, 9]:

$$\begin{aligned}\chi_{\mathbf{dis}}(v, a; \tau, z) &:= \sum_{n\in N\mathbb{Z}}(-1)^n q^{\frac{\hat{c}}{2}n^2}y^{\hat{c}n}\,\mathrm{ch}_{\mathbf{dis}}\left(\frac{v}{K}, a; \tau, z+n\tau\right)\\ &= \sum_{n\in\mathbb{Z}}\mathrm{ch}_{\mathbf{dis}}\left(\frac{v}{K}, a+Nn; \tau, z\right)\\ &= \sum_{n\in\mathbb{Z}}\frac{(yq^{Nn+a})^{\frac{v}{N}}}{1-yq^{Nn+a}}y^{2K\left(n+\frac{a}{N}\right)}q^{NK\left(n+\frac{a}{N}\right)^2}\frac{\theta_1(\tau, z)}{i\eta(\tau)^3}.\end{aligned} \quad \text{(C.15)}$$

This again corresponds to the sum of the Ramond vacuum representation with $h=\frac{\hat{c}}{8}$, $Q=\frac{v}{N}-\frac{1}{2}$ $(v=0,1,\ldots,N-1)$ over spectral flow with flow momentum m taken to be mod.N, as $m=a+N\mathbb{Z}\quad(a\in\mathbb{Z}_N)$.

The modular transformation formula can be expressed as [10, 7, 9];

$$\chi_{\mathbf{dis}}\left(v,a;-\frac{1}{\tau},\frac{z}{\tau}\right)=e^{i\pi\frac{\hat{c}}{\tau}z^2}\left[\sum_{v=0}^{N-1}\sum_{a\in\mathbb{Z}_N}\frac{1}{N}\,e^{2\pi i\frac{vv'-(v+2Ka)(v'+2Ka')}{2NK}}\,\chi_{\mathbf{dis}}(v',a';\tau,z)\right.$$
$$\left.-\frac{i}{2NK}\sum_{m'\in\mathbb{Z}_{2NK}}e^{-2\pi i\frac{(v+2Ka)m'}{2NK}}\int_{\mathbb{R}+i0}dp'\,\frac{e^{-2\pi\frac{vp'}{2NK}}}{1-e^{-\pi\frac{p'+im'}{K}}}\,\chi_{\mathbf{con}}(p',m';\tau,z)\right], \tag{C.16}$$
$$\chi_{\mathbf{dis}}\left(v,a;\tau+1,z\right)=e^{2\pi i\frac{a}{N}(v+Ka)}\,\chi_{\mathbf{dis}}\left(v,a;\tau,z\right), \tag{C.17}$$

The spectral flow property is also expressed as

$$\chi_{\mathbf{dis}}(v,a;\tau,z+r\tau+s)=(-1)^{r+s}e^{2\pi i\frac{v+2Ka}{N}s}q^{-\frac{\hat{c}}{2}r^2}y^{-\hat{c}r}\,\chi_{\mathbf{dis}}(v,a+r;\tau,z),\quad({}^{\forall}r,s\in\mathbb{Z}), \tag{C.18}$$

Modular Completion of the Irreducible Discrete Character (C.6):

$$\begin{aligned}
\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda,n;\tau,z) &:= \mathrm{ch}_{\mathbf{dis}}(\lambda,n;\tau,z)\\
&\quad -\frac{1}{2}\sum_{\nu\in\lambda+k\mathbb{Z}}\mathrm{sgn}(\nu+0)\,\mathrm{Erfc}\left(\sqrt{\frac{\pi\tau_2}{k}}\,|\nu|\right)\,q^{\frac{n^2}{k}+\frac{n}{k}\nu}\,y^{\frac{1}{k}(\nu+2n)}\,\frac{\theta_1(\tau,z)}{i\eta(\tau)^3}\\
&= \frac{\theta_1(\tau,z)}{i\eta(\tau)^3}\,y^{\frac{2n}{k}}q^{\frac{n^2}{k}}\left[\frac{(yq^n)^{\frac{\lambda}{k}}}{1-yq^n}-\frac{1}{2}\sum_{\nu\in\lambda+k\mathbb{Z}}\mathrm{sgn}(\nu+0)\,\mathrm{Erfc}\left(\sqrt{\frac{\pi\tau_2}{k}}\,|\nu|\right)\,(yq^n)^{\frac{\nu}{k}}\right]\\
&= \frac{\theta_1(\tau,z)}{2\pi\eta(\tau)^3}\,\frac{y^{\frac{2n}{k}}q^{\frac{n^2}{k}}}{1-yq^n}\sum_{\nu\in\lambda+k\mathbb{Z}}\left\{\int_{\mathbb{R}+i(k-0)}dp-\int_{\mathbb{R}-i0}dp\,(yq^n)\right\}\frac{e^{-\pi\tau_2\frac{p^2+\nu^2}{k}}\,(yq^n)^{\frac{\nu}{k}}}{p-i\nu},\\
&\qquad (0\le\lambda\le k,\quad n\in\mathbb{Z}),
\end{aligned} \tag{C.19}$$

where Erfc$(*)$ denotes the error-function defined by

$$\mathrm{Erfc}(x):=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt\,(\equiv 1-\mathrm{Erf}(x))\,. \tag{C.20}$$

The equality in the last line of (C.19) is derived from the integral formula;

$$\int_{\mathbb{R}\mp i0}dp\,\frac{e^{-\alpha p^2}}{p-i\nu}=i\pi e^{\alpha\nu^2}\mathrm{sgn}(\nu\pm 0)\mathrm{Erfc}(\sqrt{\alpha}|\nu|),\qquad(\nu\in\mathbb{R},\ \alpha>0), \tag{C.21}$$

and by using a simple contour deformation technique.

Note that $\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n; \tau, z)$ is non-holomorphic due to the explicit dependence on $\tau_2 \equiv \mathrm{Im}\tau$. It is crucial that the modular completion $\widehat{\mathrm{ch}}_{\mathbf{dis}}$ has nice modular properties. Especially, one can prove that the S-transformation formula gets considerably simplified[13];

$$\widehat{\mathrm{ch}}_{\mathbf{dis}}\left(\lambda, n; -\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{i\pi\frac{\hat{c}}{\tau}z^2}\,\frac{1}{k}\int_0^k d\lambda' \sum_{n'\in\mathbb{Z}} e^{2\pi i\frac{\lambda\lambda' - (\lambda+2n)(\lambda'+2n')}{2k}}\,\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda', n'; \tau, z). \tag{C.22}$$

It is easy to see that the T-transformation and spectral flow property are preserved by taking the completion;

$$\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n; \tau+1, z) = e^{2\pi i\frac{n}{k}(\lambda+n)}\,\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n; \tau, z), \tag{C.23}$$

$$\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n; \tau, z+r\tau+s) = (-1)^{r+s} e^{2\pi i\frac{\lambda+2n}{k}s} q^{-\frac{\hat{c}}{2}r^2} y^{-\hat{c}r}\,\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n+r; \tau, z), \quad ({}^{\forall} r, s\in\mathbb{Z}). \tag{C.24}$$

Modular Completion of the Extended Discrete Characters:

The modular completion of the discrete character $\chi_{\mathbf{dis}}$ is defined as the spectral flow sum of $\widehat{\mathrm{ch}}_{\mathbf{dis}}$ (C.19) in the similar manner to (C.15);

$$\begin{aligned}
\widehat{\chi}_{\mathbf{dis}}(v, a; \tau, z) &:= \sum_{n\in N\mathbb{Z}} (-1)^n q^{\frac{\hat{c}}{2}n^2} y^{\hat{c}n}\,\widehat{\mathrm{ch}}_{\mathbf{dis}}\left(\frac{v}{K}, a; \tau, z+n\tau\right) \\
&= \sum_{m\in\mathbb{Z}} \widehat{\mathrm{ch}}_{\mathbf{dis}}\left(\frac{v}{K}, a+Nm; \tau, z\right) \\
&= \chi_{\mathbf{dis}}(v, a; \tau, z) - \frac{1}{2}\sum_{j\in\mathbb{Z}_{2K}} R_{v+Nj, NK}(\tau)\Theta_{v+Nj+2Ka, NK}\left(\tau, \frac{2z}{N}\right)\frac{\theta_1(\tau, z)}{i\eta(\tau)^3}, \\
&= \frac{\theta_1(\tau, z)}{2\pi\eta(\tau)^3}\sum_{\substack{n\in a+N\mathbb{Z}\\ s\in v+N\mathbb{Z}}} \frac{y^{\frac{2Kn}{N}} q^{\frac{Kn^2}{N}}}{1-yq^n}\left\{\int_{\mathbb{R}+i(N-0)} dp - \int_{\mathbb{R}-i0} dp\,(yq^n)\right\}\frac{e^{-\pi\tau_2\frac{p^2+s^2}{NK}}(yq^n)^{\frac{s}{N}}}{p-is},
\end{aligned} \tag{C.25}$$

where we set

$$\begin{aligned}
R_{m,k}(\tau) &:= \sum_{\nu\in m+2k\mathbb{Z}} \mathrm{sgn}(\nu+0)\mathrm{Erfc}\left(\sqrt{\frac{\pi\tau_2}{k}}\,|\nu|\right) q^{-\frac{\nu^2}{4k}} \\
&= \frac{1}{i\pi}\sum_{\nu\in m+2k\mathbb{Z}} \int_{\mathbb{R}-i0} dp\,\frac{e^{-\pi\tau_2\frac{p^2+\nu^2}{k}}}{p-i\nu}\,q^{-\frac{\nu^2}{4k}}.
\end{aligned} \tag{C.26}$$

[13]Probably, the easiest way to prove it would be given by regarding $\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda, n; \tau, z)$ as the 'continuum limit' of $\widehat{\chi}_{\mathbf{dis}}(v, a; \tau, z)$. (See (C.27).) The modular property of $\widehat{\chi}_{\mathbf{dis}}(v, a; \tau, z)$ can be straightforwardly read off from the expansion formula of elliptic genus (4.7), and we eventually arrive at the wanted formula (C.22).

Conversely the irreducible modular completion (C.19) is reconstructed from the extended one (C.25) by taking the 'continuum limit' ;

$$\lim_{\substack{N\to\infty \\ k\equiv N/K \text{ fixed}}} \widehat{\chi}_{\mathbf{dis}}\left(v,a;\tau,z\right)=\widehat{\mathrm{ch}}_{\mathbf{dis}}\left(\lambda\equiv\frac{v}{K},a;\tau,z\right). \tag{C.27}$$

In the main text of this paper we also use an alternative notation

$$\begin{aligned}&\widehat{\boldsymbol{\chi}}(v,m;\tau,z)\equiv\widehat{\chi}_{\mathbf{dis}}(v,a;\tau,z),\quad \text{with } m\equiv v+2Ka\in\mathbb{Z}_{2NK},\quad v=0,1\ldots,N-1,\\ &\widehat{\boldsymbol{\chi}}(v,m;\tau,z)\equiv 0,\quad \text{if } m-v\notin 2K\mathbb{Z}.\end{aligned} \tag{C.28}$$

The modular transformation formulas for (C.25) are written as

$$\widehat{\chi}_{\mathbf{dis}}\left(v,a;-\frac{1}{\tau},\frac{z}{\tau}\right)=e^{i\pi\frac{\hat{c}}{\tau}z^2}\sum_{v'=0}^{N-1}\sum_{a'\in\mathbb{Z}_N}\frac{1}{N}\,e^{2\pi i\frac{vv'-(v+2Ka)(v'+2Ka')}{2NK}}\,\widehat{\chi}_{\mathbf{dis}}(v',a';\tau,z), \tag{C.29}$$

$$\widehat{\chi}_{\mathbf{dis}}\left(v,a;\tau+1,z\right)=e^{2\pi i\frac{a}{N}(v+Ka)}\,\widehat{\chi}_{\mathbf{dis}}\left(v,a;\tau,z\right). \tag{C.30}$$

Also the spectral flow property is preserved by taking the completion;

$$\widehat{\chi}_{\mathbf{dis}}(v,a;\tau,z+r\tau+s)=(-1)^{r+s}e^{2\pi i\frac{v+2Ka}{N}s}q^{-\frac{\hat{c}}{2}r^2}y^{-\hat{c}r}\,\widehat{\chi}_{\mathbf{dis}}(v,a+r;\tau,z),\quad ({}^\forall r,s\in\mathbb{Z}). \tag{C.31}$$

It may be useful to note the following identity of $R_{m,k}(\tau)$ [15], which relates the S-transformation of $R_{m,k}(\tau)$ to the Mordell integral [11, 12];

$$R_{m,k}(\tau)+\frac{i}{\sqrt{-i\tau}}\frac{1}{\sqrt{2k}}\sum_{\ell\in\mathbb{Z}_{2k}}e^{-\frac{i\pi m\ell}{k}}R_{\ell,k}\left(-\frac{1}{\tau}\right)=2ie^{-\frac{i\pi m^2\tau}{2k}}\int_{\mathbb{R}-it}dp\,\frac{e^{2\pi ik\tau p^2-2\pi m\tau p}}{1-e^{2\pi p}},\quad (0<{}^\forall t<1). \tag{C.32}$$

Witten Index :

$$\begin{aligned}&\lim_{z\to 0}\mathrm{ch}_{\mathbf{dis}}(\lambda,n;\tau,z)=\lim_{z\to 0}\widehat{\mathrm{ch}}_{\mathbf{dis}}(\lambda,n;\tau,z)=\delta_{n,0},\\ &\lim_{z\to 0}\chi_{\mathbf{dis}}(v,a;\tau,z)=\lim_{z\to 0}\widehat{\chi}_{\mathbf{dis}}(v,a;\tau,z)=\delta^{(N)}_{a,0}\equiv\begin{cases}1 & a\equiv 0\ (\mathrm{mod}\ N)\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{C.33}$$

Appendix D: Detailed Calculations for the 'Character Decomposition'

In this appendix we present a detailed calculation to derive the discrete parts of partition functions. It is almost parallel to those given in [1]. We shall only work with (3.6), and other

formulas are reproduced from it. We again assume $k = N/K$, $N = ML$ with some positive integers N, K, M, L.

Let us start with rewriting (3.6) in the form of 'Fourier transformation' (recall (3.4));

$$\widetilde{Z}^{(M)}_{\mathbf{reg}}(\tau, z\,|\alpha, \beta; \epsilon) = \frac{k}{M}\, e^{-\frac{2\pi}{\tau_2}\frac{k+4}{k}z_2^2} \sum_{w,m\in\mathbb{Z}} e^{2\pi i \frac{1}{M}(w\beta - m\alpha)}$$

$$\times \int_{\Sigma(z,\epsilon)} \frac{d^2u}{\tau_2} \left| \frac{\theta_1\left(\tau, u+\frac{k+2}{k}z\right)}{\theta_1\left(\tau, u+\frac{2}{k}z\right)}\right|^2 e^{-4\pi\frac{u_2 z_2}{\tau_2}}\, e^{-\frac{\pi k}{\tau_2}\left|u+\frac{w\tau+m}{M}\right|^2}. \tag{D.1}$$

After making a small change of integration variables, we obtain

$$\begin{aligned}\widetilde{Z}^{(M)}_{\mathbf{reg}}(\tau, z\,|\,\alpha, \beta; \epsilon) \;=\; & \frac{k}{M} e^{-\frac{2\pi}{\tau_2}z_2^2} \sum_{w,m\in\mathbb{Z}} e^{2\pi i \frac{1}{M}(w\beta - m\alpha)} \int_{\epsilon}^{1-\epsilon} ds_1 \int_0^1 ds_2 \left| \frac{\theta_1\left(\tau, -s_1\tau - s_2 + z\right)}{\theta_1\left(\tau, -s_1\tau - s_2\right)}\right|^2 \\ & \times e^{4\pi s_1 z_2}\, e^{-\frac{\pi k}{\tau_2}\left|\left(s_1+\frac{w}{M}\right)\tau + \left(s_2+\frac{m}{M}\right)+\frac{2}{k}z\right|^2}.\end{aligned}$$

By dualizing the temporal winding number m into the KK momentum n by means of the Poisson resummation formula, we can further rewrite it as

$$\begin{aligned}\widetilde{Z}^{(M)}_{\mathbf{reg}}(\tau, z\,|\alpha, \beta; \epsilon) \;=\; & \sqrt{k\tau_2}\, e^{-\frac{2\pi}{\tau_2}z_2^2} \sum_{w,n\in\mathbb{Z}} e^{2\pi i \frac{w\beta}{M}} \int_{\epsilon}^{1-\epsilon} ds_1 \int_0^1 ds_2 \left| \frac{\theta_1\left(\tau, -s_1\tau - s_2 + z\right)}{\theta_1\left(\tau, -s_1\tau - s_2\right)}\right|^2 \\ & \times e^{4\pi s_1 z_2}\, e^{-\pi\tau_2\left\{\frac{n^2}{k} + k\left(s_1+\frac{w}{M}+\frac{2z_2}{k\tau_2}\right)^2\right\} + 2\pi i n\left\{\left(s_1+\frac{w}{M}\right)\tau_1 + s_2 + \frac{2z_1}{k}\right\}} \\ =\; & \sqrt{k\tau_2}\, e^{-2\pi\frac{\hat{c}}{\tau_2}z_2^2} \sum_{w,n\in\mathbb{Z}} e^{2\pi i \frac{w\beta}{M}} \int_{\epsilon}^{1-\epsilon} ds_1 \int_0^1 ds_2 \left| \frac{\theta_1\left(\tau, -s_1\tau - s_2 + z\right)}{\theta_1\left(\tau, -s_1\tau - s_2\right)}\right|^2 \\ & \times e^{-\pi\tau_2\left\{\frac{1}{k}(Mn+\alpha)^2 + k\left(s_1+\frac{w}{M}\right)^2\right\} + 2\pi i (Mn+\alpha)\left\{\left(s_1+\frac{w}{M}\right)\tau_1 + s_2\right\}}\, y^{\frac{w}{M}+\frac{Mn+\alpha}{k}}\, \bar{y}^{\frac{w}{M}-\frac{Mn+\alpha}{k}}.\end{aligned} \tag{D.2}$$

Substituting the identity[14]; ($u \equiv s_1\tau + s_2$, $0 < s_1 < 1$);

$$\begin{aligned}\left| \frac{\theta_1\left(\tau, -s_1\tau - s_2 + z\right)}{\theta_1\left(\tau, -s_1\tau - s_2\right)}\right|^2 = & \left|\frac{\theta_1(\tau, z)}{i\eta(\tau)^3}\right|^2 \sum_{\ell,\tilde{\ell}\in\mathbb{Z}} \frac{yq^{\ell}}{1-yq^{\ell}} \cdot \left[\frac{yq^{\tilde{\ell}}}{1-yq^{\tilde{\ell}}}\right]^* \\ & \times e^{-2\pi i (s_1\tau_1+s_2)(\ell-\tilde{\ell}) + 2\pi s_1\tau_2(\ell+\tilde{\ell})}.\end{aligned} \tag{D.3}$$

into (D.2), one can easily integrate s_2 out, which just yields the constraint

$$Mn + \alpha = \ell - \tilde{\ell}. \tag{D.4}$$

We next evaluate the s_1-integral. Picking up relevant terms, we obtain

$$e^{-\pi\tau_2\frac{N}{K}s_1^2 - 2\pi s_1\left\{\tau_2\frac{L}{K}w - i\tau_1(Mn+\alpha) + i\tau_1(\ell-\tilde{\ell}) - \tau_2(\ell+\tilde{\ell})\right\}} = e^{-\pi\tau_2\frac{N}{K}s_1^2 - 2\pi s_1\tau_2\frac{v}{K}}, \tag{D.5}$$

[14]See *e.g.* [1].

where we set

$$v := Lw - K(\ell + \tilde{\ell}), \tag{D.6}$$

and used the condition (D.4). Utilizing a Gaussian integral, we obtain

$$\begin{aligned}
\int_{\epsilon}^{1-\epsilon} ds_1 \, e^{-\pi\tau_2 \frac{N}{K} s_1^2 - 2\pi s_1 \tau_2 \frac{v}{K}} &= \sqrt{\frac{\tau_2}{NK}} \int_{\epsilon}^{1-\epsilon} ds_1 \int_{\mathbb{R}-i0} dp \, e^{-\frac{\pi}{NK}\tau_2 p^2 - 2\pi i \tau_2 \frac{s_1}{K}(p-iv)} \\
&= \sqrt{\frac{K}{N\tau_2}} \frac{1}{2\pi i} \int_{\mathbb{R}-i0} dp \, \frac{e^{-\frac{\pi}{NK}\tau_2 p^2}}{p-iv} \left\{ e^{-\varepsilon(v+ip)} - e^{\varepsilon(v+ip)} e^{-2\pi i \tau_2 \frac{1}{K}(p-iv)} \right\},
\end{aligned} \tag{D.7}$$

where we set $\varepsilon \equiv 2\pi \frac{\tau_2}{K}\epsilon \, (> 0)$.

Collecting remaining exponents of q and y, we further obtain the factor;

$$e^{-\pi\tau_2 \frac{v^2}{NK}} \, q^{\frac{1}{N}\left(K\ell^2 + \ell v\right)} \bar{q}^{\frac{1}{N}\left(K\tilde{\ell}^2 + \tilde{\ell} v\right)} \, y^{\frac{2K}{N}\left(\ell + \frac{v}{2K}\right)} \bar{y}^{\frac{2K}{N}\left(\tilde{\ell} + \frac{v}{2K}\right)}.$$

Combining all the pieces we finally obtain

$$\begin{aligned}
\widetilde{Z}_{\mathbf{reg}}^{(M)}(\tau, z \,|\, \alpha, \beta\,; \epsilon) &= e^{-2\pi \frac{\hat{c}}{\tau_2} z_2^2} \left| \frac{\theta_1(\tau, z)}{\eta(\tau)^3} \right|^2 \sum_{\substack{\ell, \tilde{\ell} \in \mathbb{Z} \\ \ell - \tilde{\ell} \in \alpha + M\mathbb{Z}}} \; \sum_{\substack{v \in \mathbb{Z} \\ v + K(\ell + \tilde{\ell}) \in L\mathbb{Z}}} \\
&\times \frac{1}{2\pi i} \left[\int_{\mathbb{R}-i0} dp \, e^{-\varepsilon(v+ip)} y q^{\ell} \left[y q^{\tilde{\ell}} \right]^* - \int_{\mathbb{R}+i(N-0)} dp \, e^{\varepsilon(v+ip)} \right] \\
&\times e^{2\pi i \frac{\beta}{N}\left\{v + K(\ell + \tilde{\ell})\right\}} \, \frac{e^{-\pi\tau_2 \frac{p^2+v^2}{NK}}}{p - iv} \, \frac{(yq^{\ell})^{\frac{v}{N}}}{1 - yq^{\ell}} \left[\frac{(yq^{\tilde{\ell}})^{\frac{v}{N}}}{1 - yq^{\tilde{\ell}}} \right]^* y^{\frac{2K}{N}\ell} q^{\frac{K}{N}\ell^2} \left[y^{\frac{2K}{N}\tilde{\ell}} q^{\frac{K}{N}\tilde{\ell}^2} \right]^* .
\end{aligned} \tag{D.8}$$

Here the factor

$$y q^{\ell} \cdot \left[y q^{\tilde{\ell}} \right]^* \cdot e^{-2\pi i \tau_2 \frac{1}{K}(p - iv)}$$

was absorbed into the change of variables $p =: p' - iN$, $v =: v' - N$ with

$$\begin{aligned}
&\frac{p^2 + v^2}{NK} = \frac{p'^2 + v'^2}{NK} - \frac{2i}{K}(p' - iv'), \qquad p - iv = p' - iv', \\
&\left(yq^{\ell}\right)^{\frac{v}{N}} \left[\left(y q^{\tilde{\ell}} \right)^{\frac{v}{N}} \right]^* = \left(yq^{\ell}\right)^{\frac{v'}{N} - 1} \left[\left(y q^{\tilde{\ell}} \right)^{\frac{v'}{N} - 1} \right]^* .
\end{aligned}$$

At this stage, one can successfully extract a sesquilinear form of the modular completion $\widehat{\chi}_{\mathbf{dis}}$ from (D.8) as the discrete part in a similar manner to (2.25), that is,

$$\widetilde{Z}_{\mathbf{reg}}^{(M)}(\tau, z \,|\, \alpha, \beta; \epsilon) = \widetilde{Z}_{\mathbf{dis}}^{(M)}(\tau, z \,|\, \alpha, \beta) + [\text{sesquilinear form of } \chi_{\mathbf{con}}(\tau, z)].$$

Performing a ‘completion of the square’ by using the last line of (C.25), one can achieve

$$\widetilde{Z}_{\mathbf{dis}}^{(M)}(\tau, z \,|\, \alpha, \beta) = e^{-2\pi \frac{\hat{c}}{\tau_2} z_2^2} \sum_{(v, a, \tilde{a}) \in \mathcal{R}(M, \alpha)} e^{2\pi i \frac{\beta}{N}\{v + K(a + \tilde{a})\}} \, \widehat{\chi}_{\mathbf{dis}}(v, a; \tau, z) \left[\widehat{\chi}_{\mathbf{dis}}(v, \tilde{a}; \tau, z) \right]^* , \tag{D.9}$$

where the range of summation $\mathcal{R}(M,\alpha)$ is given in (3.32), namely,

$$\begin{aligned}\mathcal{R}(M,\alpha) \quad = \quad & \{(v,a,\tilde{a}) \in \mathbb{Z}\times\mathbb{Z}_N\times\mathbb{Z}_N\ ;\ 0 \le v \le N-1, \\ & \qquad\qquad a-\tilde{a} \equiv \alpha \pmod{M},\ v+K(a+\tilde{a}) \in L\mathbb{Z}\}\,.\end{aligned}$$

This is the desired formula (3.18).

References

[1] T. Eguchi and Y. Sugawara, JHEP **1103**, 107 (2011) [arXiv:1012.5721 [hep-th]].

[2] E. Witten, Phys. Rev. **D44** (1991) 314; G. Mandal, A. Sengupta and S. Wadia, Mod. Phys. Lett. **A6** (1991) 1685; I. Bars and D. Nemeschansky, Nucl. Phys. **B348** (1991) 89; S. Elizur, A. Forge and E. Rabinovici, Nucl. Phys. **B359** (1991) 581; R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. B **371**, 269 (1992).

[3] J. Troost, JHEP **1006**, 104 (2010) [arXiv:1004.3649 [hep-th]].

[4] S. K. Ashok, J. Troost, JHEP **1103**, 067 (2011). [arXiv:1101.1059 [hep-th]].

[5] A. Hanany, N. Prezas and J. Troost, JHEP **0204**, 014 (2002) [arXiv:hep-th/0202129].

[6] D. Israel, A. Pakman and J. Troost, JHEP **0404**, 043 (2004) [arXiv:hep-th/0402085].

[7] T. Eguchi and Y. Sugawara, JHEP **0405**, 014 (2004) [arXiv:hep-th/0403193].

[8] D. Israel, C. Kounnas, A. Pakman and J. Troost, JHEP **0406**, 033 (2004) [arXiv:hep-th/0403237].

[9] T. Eguchi and Y. Sugawara, JHEP **0501**, 027 (2005) [arXiv:hep-th/0411041].

[10] T. Eguchi and Y. Sugawara, JHEP **0401**, 025 (2004) [arXiv:hep-th/0311141].

[11] L.J. Mordell, Quarterly Journal of Math. **68**, 329 (1920).

[12] G.N. Watson, J. London Math. Soc. **11**, 55 (1936).

[13] T. Eguchi and A. Taormina, Phys. Lett. B **200**, 315 (1988); Phys. Lett. B **210**, 125 (1988).

[14] M. Eichler and D. Zagier, *"The Theory of Jacobi Forms,"* (Birkhäuser, 1985).

[15] S. Zwegers, PhD thesis, "Mock Theta functions", Utrecht University, 2002.

[16] K. Bringman and K. Ono, "Lifting cusp forms to Maass forms with an application to partitions", Proceedings of the National Academy of Sciences of the United States of America **104** (10) 3725-3731.

[17] T. Eguchi and K. Hikami, J. Phys. A **42**, 304010 (2009) [arXiv:0812.1151 [math-ph]]; T. Eguchi and K. Hikami, arXiv:0904.0911 [math-ph].

[18] V. A. Fateev, A. B. Zamolodchikov, Sov. Phys. JETP **62**, 215-225 (1985).

[19] D. Gepner and Z. A. Qiu, Nucl. Phys. B **285**, 423 (1987).

[20] A. Giveon, M. Porrati, E. Rabinovici, Phys. Rept. **244**, 77-202 (1994). [hep-th/9401139], and references therein.

[21] Y. Kazama and H. Suzuki, Nucl. Phys. B **321**, 232 (1989).

[22] D. Karabali and H. J. Schnitzer, Nucl. Phys. B **329**, 649 (1990).

[23] K. Gawedzki and A. Kupiainen, Nucl. Phys. B **320**, 625 (1989).

[24] H. J. Schnitzer, Nucl. Phys. B **324**, 412 (1989).

[25] J. M. Maldacena, G. W. Moore and N. Seiberg, JHEP **0107**, 046 (2001) [arXiv:hep-th/0105038].

[26] E. Witten, Commun. Math. Phys. **109**, 525 (1987).

[27] A. Polishchuk, arXiv:math.AG/9810084; A. M. Semikhatov, A. Taormina and I. Y. Tipunin, arXiv:math.qa/0311314.

[28] K. Gawedzki, arXiv:hep-th/9110076.

[29] W. Boucher, D. Friedan and A. Kent, Phys. Lett. B **172**, 316 (1986),

[30] V. K. Dobrev, Phys. Lett. B **186**, 43 (1987); E. Kiritsis, Int. J. Mod. Phys. A **3**, 1871 (1988).

[31] L. J. Dixon, M. E. Peskin and J. Lykken, Nucl. Phys. B **325**, 329 (1989).

[32] A. Giveon and D. Kutasov, JHEP **9910**, 034 (1999) [arXiv:hep-th/9909110]; A. Giveon and D. Kutasov, JHEP **0001**, 023 (2000) [arXiv:hep-th/9911039].

[33] J. M. Maldacena and H. Ooguri, J. Math. Phys. **42** (2001) 2929, hep-th/0001053.